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CONEXIONES DE AMBROSE-SINGER Y ESTRUCTURAS HOMOGÉNEAS EN VARIEDADES PSEUDO-RIEMANNIANAS

AMBROSE-SINGER CONNECTIONS AND HOMOGENEOUS STRUCTURES ON PSEUDO-RIEMANNIAN MANIFOLDS

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PRESENTADA POR

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Memoria presentada para optar al grado de
Doctor en Ciencias Matemáticas por

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A mis padres

"At ubi materia, ibi Geometria."
"Where there is matter, there is geometry"
Johannes Kepler

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Summary

Ambrose-Singer connections and homogeneous structures on pseudo-Riemannian manifolds.
Conexiones de Ambrose-Singer y estructuras homogéneas en variedades pseudo-Riemannianas.

Introduction

Homogeneous and locally homogeneous spaces enjoy a large group of internal symmetries. For that reason they constitute a distinguished class of spaces on which the study of pseudo-Riemannian geometry is especially rich and varied. This kind of spaces have been extensively studied by means of many different methods and techniques. One difficulty arising is that the same pseudo-Riemannian manifold (M, g) can admit several different descriptions as a coset space G/H. It is surprising how little is known about this problem for many well-known spaces. One of the most fruitful approaches was attempted by Ambrose and Singer, who in 1958 [4] extended Cartan's characterization of symmetric spaces. They characterized connected, simply-connected and complete homogeneous Riemannian manifolds as Riemannian manifolds (M, g) admitting a linear connection $\widetilde{\nabla}$ satisfying

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} S = 0,$$

where $S = \nabla - \widetilde{\nabla}$, ∇ is the Levi-Civita connection of g, and R is the curvature of g. These equations would become later known as Ambrose-Singer equations, and connections satisfying them as *Ambrose-Singer connections.* The previous result not only characterizes homogeneous spaces in a "nice" way, but also introduces a new tool for studying the geometry of this kind of manifolds, namely the Ambrose-Singer connection $\widetilde{\nabla}$ and the so called *homogeneous structure tensor* S (or *homogeneous structure* for short). Since their introduction, these objects have proved to be very useful, probably due to the combination of their geometric and algebraic natures. The first results in this direction were obtained by Tricerri and Vanhecke in [60], where the algebraic nature of S allowed the authors to achieved a classification of homogeneous Riemannian structure tensors into eight different classes using only algebraic arguments and a representation theoretical approach. They further identified two of those classes with spaces of constant sectional curvature and naturally reductive homogeneous spaces respectively. This theory was extended to locally homogeneous Riemannian spaces by several authors (see for instance [40] and [59]). In this setting, the canonical Ambrose-Singer connection constructed by Kowalski in [40] becomes the central axis around which the theory is built.

Ambrose-Singer Theorem was generalized by Kiričenko to the case when the Riemannian manifold (M, g) is endowed with a geometric structure defined by a set of tensor fields $P_1, \ldots, P_n$. In that case, one have to add the conditions

$$\widetilde{\nabla} P_1 = 0, \ldots, \widetilde{\nabla} P_n = 0$$

to Ambrose-Singer equations. Similar classifications of homogeneous Riemannian structures to that provided by Tricerri and Vanhecke were obtained by several authors in the presence of different geometric structures, and in [26], the classification for all the holonomies appearing in Berger's list is achieved by a representation theoretical approach. In many cases (such as Kähler, hyper-Kähler, quaternion Kähler, G_2 or Spin(7)) these classifications contain a class consisting of sections of a bundle whose rank grows linearly with the dimension of the manifold. For that reason homogeneous structures belonging to these classes are called of *linear type*. The corresponding tensor fields S are defined by a set of vector fields satisfying a system of PDE's equivalent to Ambrose-Singer equations. The importance of this kind of structures relies on the fact that in the purely Riemannian case so as in the case of Kähler and quaternion Kähler manifolds, homogeneous structures of linear type characterize spaces of negative constant sectional (resp. holomorphic sectional, quaternionic sectional) curvature (see [60], [29] and [16]).

In [31] Ambrose-Singer Theorem is adapted to metrics with arbitrary signature. As it is well known, every homogeneous Riemannian manifold is reductive, but this is no longer true if the metric is not definite. This way, the pseudo-Riemannian version of Ambrose-Singer Theorem states that the existence of an Ambrose-Singer connection characterizes reductive homogeneous pseudo-Riemannian manifolds under suitable topological conditions.

Some of the techniques used in the Riemannian case have also been adapted to metrics with signatures in order to obtain classifications of homogeneous pseudo-Riemannian structures, both in the purely pseudo-Riemannian case and in the presence of a geometric structure (see for instance [31], [30] and [8]). In this situation we also have that in many cases (such as pseudo-Kähler, para-Kähler, pseudo-hyper-Kähler, para-hyper-Kähler, pseudo-quaternion Kähler, para-quaternion Kähler, $G^*_{2(2)}$ or Spin(4,3)) there is a class, also called of linear type, consisting of sections of a bundle whose rank grows linearly with the dimension of the manifold. When metrics with signature are studied, the causal character of the vector fields defining the homogeneous structure tensor needs to be taken into account. In the purely pseudo-Riemannian case, non-degenerate homogeneous structures of linear type (i.e. given by a non null vector field) characterize spaces of constant sectional curvature [31]. On the other hand, degenerate homogeneous structures of linear type (i.e. given by a null vector field) characterize singular scale-invariant homogeneous plane waves [46]. Furthermore, in [45] it is shown that homogeneous structures in the composed class $\mathcal{S}_1 + \mathcal{S}_3$ are related to a larger class of singular homogeneous plane waves. It is worth pointing out that very less is known about homogeneous spaces with holonomy $G^*_{2(2)}$ or $Spin(4,3)$. It is even very difficult to find non-flat examples in the literature, and most of them have low dimensional holonomy. Concerning this situation, new examples of Lie groups with left-invariant metrics with full holonomy $G^*_{2(2)}$ have been recently obtained in [27].

Objectives

The present dissertation has three main objectives. In the first place, we want to extend Ambrose-Singer Theorem to locally homogeneous pseudo-Riemannian manifolds. The adaptation of the theory is not as straightforward as in the Riemannian case, and new concepts need to be developed. In addition we will like to explore how the construction of the canonical connection made by Kowalski fits in the pseudo-Riemannian setting, and in particular if the reconstruction of a locally homogeneous space from the curvature and its covariant derivatives up to finite order still holds. Secondly, we would like to characterize homogeneous structures of linear type in the pseudo-Kähler, para-Kähler, pseudo-quaternion Kähler and para-quaternion Kähler cases. As it happened in the purely pseudo-Riemannian case, the causal character of the vector fields defining the homogeneous structure opens room for new objects and scenarios, which do not exist in the Riemannian realm. Finally, we are interested in studying the behavior of homogeneous structures within the framework of reduction under a group of isometries. Reduction procedures are widely use in many settings of differential geometry in order to construct new objects or to obtain new information from known cases. This way, a reduction scheme for homogeneous structures can be very convenient for the study of homogeneity.

Outline and results

In Chapter 1 we settle the foundations for subsequent chapters. More precisely, we briefly introduce the theory of principal bundles and connections. We also define the holonomy of a connection and present the most relevant results. This concept will be central throughout the rest of the manuscript. We then apply this theory to pseudo-Riemannian manifolds and G-structures. After stating Berger's Theorem, we describe some basic features of the geometric structures we will deal with in Chapters 5, 6 and 7. We finally introduce the basics about homogeneous spaces and define the canonical connection associated to a reductive homogeneous space. This is the first and most representative example of Ambrose-Singer connection.

In Chapter 2 we state Ambrose-Singer Theorem, which is the starting point of the theory of Ambrose-Singer connections and of the present dissertation. We first contextualize this result relating it with Cartan's characterization of symmetric spaces. We also present Kiričenko's Theorem, which extends Ambrose-Singer Theorem to the case when the manifold is endowed with an extra geometric structure. Since the author of this thesis has not found any proof of Kiričenko's Theorem in the literature, an original proof is given. We end this chapter defining the homogeneous structure tensor associated to an Ambrose-Singer connection. We introduce the corresponding infinitesimal model, Nomizu construction and transvection algebra, and discuss some of their properties.

In Chapter 3 we develop the theory of Ambrose-Singer connections on locally homogeneous pseudo-Riemannian manifolds. We prove that a locally homogeneous pseudo-Riemannian manifold admits an Ambrose-Singer connection if it satisfies an algebraic condition concerning the set of local Killing vector fields (Theorem 3.1.9). In analogy with the global case, we call this condition *reductive*. Conversely, we show that a pseudo-Riemannian manifold admitting an Ambrose-Singer connection is locally homogeneous and reductive (Theorem 3.1.10). As it is well known, different Lie (pseudo-)groups can act transitively on the same (locally) homogeneous manifold. We will see that the notion of reductivity is a property concerning the action of a certain Lie (pseudo-)group rather than a property of the manifold itself. Note that this follows the spirit of F. Klein's Erlangen Programm, pointing out that different actions on the same manifold might have a very different nature. Several examples will explore the possible scenarios. We will further extend the previous results to manifolds endowed with an extra geometric structure (Theorems 3.1.16 and 3.1.17). Following the work of Kowalski in the Riemannian case [40], a new condition, which we will call *strong reductivity*, will naturally appear. We prove that strongly reductive (locally) homogeneous pseudo-Riemannian manifolds admit a very special Ambrose-Singer connection analogous to the canonical connection constructed by Kowalski in the Riemannian case (Theorem 3.2.8). Unlike the reductivity condition, this new condition is indeed a property of the manifold itself and not of the action of any Lie (pseudo-)group. We study some properties of strongly reductive locally homogeneous pseudo-Riemannian manifolds, and in particular we see (Theorem 3.3.2) that they can be recovered from the curvature and its covariant derivatives up to finite order at some point (recall that this property is enjoyed by all locally homogeneous Riemannain manifolds [49]).

In Chapter 4 we exploit the algebraic nature of homogeneous structures in order to obtain classification results. We first sketch a general procedure to classify homogeneous structures with or without the presence of a geometric structure, and we then specify it for the geometric structures and the holonomies appearing in Chapters 5, 6 and 7. We define the so called *homogeneous structures of linear type*, which are homogeneous structures characterized by a set of vector fields. These will be the main object of study in Chapters 5 and 6. All the classifications obtained in this chapter were previously obtained by several authors, except for the para-quaternion Kähler, pseudo-hyper-Kähler and para-hyper-Kähler cases which are original.

In Chapter 5 we study homogeneous pseudo-Kähler and para-Kähler structures of linear type. On the one hand we prove that pseudo-Kähler and para-Kähler manifolds admitting a non-degenerate homogeneous pseudo-Kähler and para-Kähler structures of linear type respectively (see Definitions 4.2.6 and 4.2.9) have constant holomorphic and para-holomorphic sectional curvature respectively (Theorem 5.1.2). We moreover show that the corresponding complex and para-complex space forms only admits these kind of structures locally, unless the metric is definite (Theorems 5.3.1). On the other hand we obtain the holonomy (Propositions 5.2.2 and 5.2.4) and the local form of pseudo-Kähler and para-Kähler manifolds admitting a degenerate homogeneous pseudo-Kähler and para-Kähler structures of linear type respectively (Propositions 5.2.5 and 5.2.6), focusing on the singular nature of the metrics. We compute the associated infinitesimal model and transvection algebra for both cases, and study the completeness of the corresponding homogeneous model (Theorem 5.3.2). We finally exhibit the relation between degenerate structures and certain kind of homogeneous plane waves. Some of the results contained in this chapter, namely those referring to strongly degenerate structures, were published in [18].

In Chapter 6 we study homogeneous pseudo-quaternion Kähler and para-quaternion Kähler structures of linear type. On the one hand we prove that pseudo-quaternion Kähler and para-quaternion Kähler manifolds admitting a non-degenerate homogeneous pseudo-quaternion Kähler and para-quaternion Kähler structures of linear type respectively (see Definitions 4.2.12 and 4.2.15) have constant quaternionic and para-quaternionic sectional curvature respectively (Theorem 6.1.1). We moreover show that the corresponding quaternion and para-quaternion space forms only admits these kind of structures locally, unless the metric is definite (Theorem 6.2.1). On the other hand we show that pseudo-quaternion Kähler and para-quaternion Kähler manifolds admitting a degenerate homogeneous pseudo-quaternion Kähler and para-quaternion Kähler structures of linear type are flat (Theorem 6.1.1). We compute the associated infinitesimal model and transvection algebra for the non-degenerate case, and study the completeness of the corresponding homogeneous model (Theorem 6.2.2).

Finally, in Chapter 7 we study homogeneous structures within the framework of reduction under a group H of isometries. In a first result, H is a normal subgroup of the group of symmetries associated to a homogeneous structure $\bar{S}$ defined on a globally homogeneous space. In this case $\bar{S}$ can be reduced to a homogeneous structure in the space of orbits under the action of H (Theorem 7.1.4). In a second result we study under which conditions a homogeneous structure $\bar{S}$ defined on the total space of a principal bundle $\pi : (\bar{M}, \bar{g}) \to (M, g)$ reduces to a homogeneous structure on the base space (M, g). The answer (Theorem 7.2.1) involves an additional condition on the so called *mechanical connection* of the principal bundle which resembles to the extra equation appearing in Kiričenko's Theorem. The behavior of the classes of homogeneous tensors are also investigated when reduction is performed (Proposition 7.2.3). It turns out that the geometry of the fibres is involved in the preservation of some of them (Proposition 7.2.4). Some classical examples illustrate the theory. Finally, the reduction procedure is applied to fiberings of almost contact manifolds over almost Hermitian manifolds. If the homogeneous structure is moreover cosymplectic or Sasakian, the obtained reduced homogeneous structure is pseudo-Kähler. We will use this result to obtain some properties of homogeneous cosymplectic and Sasakian structures of linear type (Propositions 7.3.8 and 7.3.10). The contents of this chapter are included in [19].

Conclusions

Regarding the first objective, we have been able to extend Ambrose-Singer Theorem and the theory of Ambrose-Singer connections to locally homogeneous pseudo-Riemannian manifolds. We have seen that the arguments used in the Riemannian case do not directly hold in this setting, and a deeper inspection has led to new concepts as reductive and strong reductive locally homogeneous pseudo-Riemannian manifolds. These new concepts reveal themselves as necessary not only in order to develop the theory of Ambrose-Singer connections, but to extend the results from the Riemannian setting to metrics of arbitrary signature. It is very interesting how the transition from the Riemannian setting to the case of metrics with signature usually gives some perspective and reveals the Riemannian case as a very special situation. Regarding this, it is worth pointing out that very less is known about non-reductive homogeneous pseudo-Riemannian manifolds (see for example [25]). I believe that Chapter 3 sheds some light to this problem.

Concerning the second objective, we have been able to characterize the class of homogeneous structures of linear type in the case of pseudo-Kähler, para-Kähler, pseudo-quaternion Kähler and para-quaternion Kähler geometries. It is very interesting how the causal character of the set of vector field defining the homogeneous structures separates two different worlds. On the one hand, non-degenerates structures give results which resemble to the Riemannian case, namely they characterize manifolds of constant (para-)holomorphic and (para-)quaternionic sectional curvature. On the other hand, degenerate structures have no Riemannian counterpart, so that genuine pseudo-Riemannian situations appear. More precisely, in the pseudo-Kähler and para-Kähler cases, these kind of homogeneous structures characterize manifolds whose underlying geometry can be interpreted as a (para-)complex generalization of the geometry of homogeneous plane waves. In the pseudo-quaternion and para-quaternion cases the condition $\widetilde{\nabla}R = 0$ in Ambrose-Singer equations becomes too strong, and the manifold is forced to be flat, suggesting that the notion of homogeneous plane wave cannot be extended to geometries of quaternionic type. Another remarkable feature about both degenerate and non-degenerate structures, is that completeness issues naturally arise. More precisely, besides the fact that the underlying geometry of degenerate structures seems to be singular (in a cosmological sense), all homogeneous models associated to these homogeneous structures are necessarily incomplete. The origin of these completeness issues seems fuzzy at this moment and could be an interesting topic of research in the future.

Finally, the third objective has been accomplished providing a good reduction scheme for homogeneous structures. It is noteworthy the role played by the geometry of the principal fiber bundle and the Kiričenko condition on the mechanical connection involved in the reduction scheme. It is also worth stressing how the geometry of the fibers are involved in the preservation of some classes of homogeneous structures. As desired, this results have allowed us to study new objects from known ones. More precisely, we have been able to study some properties of homogeneous cosymplectic and Sasakian structures of linear type making use of the reduction procedure and our knowledge of homogeneous pseudo-Kähler structures of linear type. Many problems still remain open for the future. In the first place, an inverse procedure would be of great value, namely we would like to find suitable conditions to define homogeneous structures on the total space of a principal bundle from homogeneous structures defined in the base space. Secondly, a complete study of homogeneous cosymplectic and Sasakian structures of linear type remains to be done, since we have only deal with invariant structures. The properties we have obtained in this part of the thesis would be a head start for this purpose.

The works derived from topics of this thesis appear in [18], [19], [20], [27], [41], and [42].

Resumen

Conexiones de Ambrose-Singer y estructuras homogéneas en variedades pseudo-Riemannianas / Ambrose-Singer connections and homogeneous structures on pseudo-Riemannian manifolds.

Introducción

Los espacios homogéneos y localmente homogéneos poseen un gran grupo de simetrías internas. Por esta razón constituyen una clase distinguida de espacios en los cuales el estudio de la geometría pseudo-Riemanniana es especialmente rica y variada. Este tipo de espacios han sido extensamente estudiados por medio de diferentes métodos y técnicas. Una de las dificultades que aparecen es que la misma variedad pseudo-Riemanniana (M, g) puede admitir diferentes descripciones como espacio cociente G/H. Es sorprendente lo poco que se sabe acerca de este problema incluso para espacios bien conocidos. Una de las aproximaciones más fructíferas fue desarrollada por Ambrose y Singer, quienes en 1958 [4] extendieron la caracterización de espacios simétricos dada por Cartan. Estos autores caracterizaron los espacios homogéneos Riemannianos conexos, simplemente conexos y completos como aquellos que admiten una conexión lineal $\widetilde{\nabla}$ que satisface

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} S = 0,$$

donde $S = \nabla - \widetilde{\nabla}$, ∇ es la conexión de Levi-Civita de g, y R es la curvatura de g. Estas ecuaciones pasaron a llamarse ecuaciones de Ambrose-Singer, y las conexiones satisfaciéndolas pasaron a llamarse *conexiones de Ambrose-Singer*. El resultado anterior no sólo caracteriza los espacios homogéneos de una manera "agradable", sino que además introduce una nueva herramienta para su estudio: las conexiones de Ambrose-Singer y el llamado *tensor de estructura homogénea* S (o *estructura homogénea* para abreviar). Desde su introducción, estos objetos han probado ser de gran utilidad, probablemente debido a la combinación de sus naturalezas algebraica y geométrica. El primer resultado en esta dirección fue obtenido por Tricerri y Vanhecke en [60], donde la naturaleza algebraica de S permitió a los autores dar una clasificación de los tensores estructura homogénea Riemannianos usando solamente argumentos algebraicos y Teoría de Representación. Más aún, los autores identificaron dos de las clases con los espacios de curvatura seccional constante y los espacios naturalmente reductivos respectivamente. Esta teoría fue extendida a espacios Riemannianos localmente homogéneos por varios autores (véase por ejemplo [40] y [59]). En este ámbito, la conexión canónica construida por Kowalski en [40] es el eje central alrededor del cual se articula la teoría.

El Teorema de Ambrose-Singer fue generalizado por Kiričenko al caso en que la variedad Riemanniana (M, g) estuviese dotada de una estructura geométrica definida por un conjunto de campos tensoriales $P_1, \ldots, P_n$. En este caso tenemos que añadir las condiciones $\widetilde{\nabla} P_1 = 0, \ldots, \widetilde{\nabla} P_n = 0$ a las ecuaciones de Ambrose-Singer. Desde entonces varios autores han obtenido clasificaciones similares a la dada por Tricerri y Vanhecke en presencia de una estructura geométrica, y en [26], se estudian las clasificaciones para todas las holonomías Riemannianas en la lista de Berger por medio de Teoría de Representación. En muchos casos (tales como Kähler, hiper-Kähler, cuaterniónico Kähler, G_2 o Spin(7)) estas clasificaciones contienen una clase que consiste en secciones de un fibrado cuyo rango crece linealmente con la dimensión de la variedad. Por esta razón, las estructuras homogéneas que pertenecen a estas clases son llamadas *de tipo* *lineal*. Los tensores S correspondientes están definidos por un conjunto de campos vectoriales que satisfacen un sistema de EDPs equivalente a las ecuaciones de Ambrose-Singer. La importancia de este tipo de estructuras radica en el hecho de que en los caso puramente Riemanniano, Kähler y cuaterniónico Kähler, caracterizan respectivamente espacios de curvatura seccional, curvatura seccional holomorfa, y curvatura seccional quaterniónica constante negativa (véase [60], [29] y [16]).

En [31] el Teorema de Ambrose-Singer es adaptado a métricas con signatura arbitraria. Como es bien conocido, todo espacio homogéneo Riemanniano es reductivo, pero esto no es cierto si la métrica no es definida. De esta forma, la versión pseudo-Riemanniana del Teorema de Amgrose-Singer establece que la existencia de una conexión de Ambrose-Singer caracteriza los espacios homogéneos pseudo-Riemannianos reductivos bajo ciertas condiciones topológicas. Algunas de las técnicas utilizadas en el caso Riemanniano han sido adaptadas a métricas con signatura para obtener resultados de clasificación, tanto en presencia de una estructura geométrica como en su ausencia (véase por ejemplo [31], [30] y [8]). En esta situación también se tienen en muchos casos (tales como pseudo-Kähler, para-Kähler, pseudo-hiper-Kähler, para-hiper-Kähler, pseudo-cuaterniónico Kähler, para-cuaterniónico Kähler, $G^*_{2(2)}$ o $\mathrm{Spin}(4,3)$) clases que consisten en secciones de un fibrado cuyo rango crece linealmente con la dimensión de la variedad (llamadas igualmente de tipo lineal). Cuando se estudian métricas con signatura el carácter causal de los campos vectoriales que definen un tensor estructura homogénea de tipo lineal juega un papel importante. En el caso puramente pseudo-Riemanniano, las estructuras homogéneas de tipo lineal no degeneradas (dadas por un campo vectorial no isótropo) caracterizan espacios de curvatura seccional constante [31]. Por otro lado, las estructuras homogéneas de tipo lineal degeneradas (dadas por un campo vectorial isótropo) caracterizan las llamadas "singular scale-invariant homogeneous plane waves" [46]. Más aún, en [45] se prueba que las estructuras homogéneas en la clase compuesta $\mathcal{S}_1 + \mathcal{S}_3$ están relacionadas con una clase más amplia de ondas planas singulares. Merece la pena señalar lo poco que se sabe sobre espacios homogéneos con holonomía $G^*_{2(2)}$ o $Spin(4,3)$. Es muy difícil encontrar ejemplos en la literatura que no sean planos, y la mayoría de ellos tienen una holonomía de muy baja dimensión. Acerca de este problema, nuevos ejemplos de grupos de Lie con métricas invariantes y holonomía igual a $G^*_{2(2)}$ han sido obtenidos recientemente en [27].

Objetivos

La presente tesis doctoral tiene tres objetivos principales. En primer lugar queremos extender el Teorema de Ambrose-Singer a variedades localmente homogéneas pseudo-Riemannianas. La adaptación de la teoría desde el caso global al caso local no es tan directa como en el caso Riemanniano, por lo que es necesario desarrollar nuevos conceptos. Además nos gustaría explorar cómo encaja la construcción de la conexión canónica de Kowalski en el ámbito pseudo-Riemanniano, y en particular si se mantiene la reconstrucción de una variedad localmente homogénea a partir de su curvatura y sus derivadas covariantes hasta orden finito en un punto. En segundo lugar queremos caracterizar las estructuras homogéneas de tipo lineal en los casos pseudo-Kähler, para-Kähler, pseudo-cuaterniónico Kähler y para-cuaterniónico Kähler. Como ocurre en el caso puramente pseudo-Riemanniano, el carácter causal de los campos vectoriales que definen la estructura homogénea en cuestión abre espacio para nuevos objetos y escenarios que no existen en la categoría Riemanniana. Finalmente estamos interesados en estudiar el comportamiento de las estructuras homogéneas en el marco de la reducción bajo la acción de un grupo de isometrías. Los procesos de reducción son ampliamente usados en diferentes ramas de la Geometría Diferencial con el objetivo de construir nuevos objetos u obtener nueva información a partir de casos conocidos. De esta forma un esquema de reducción para las estructuras homogéneas puede ser muy conveniente para estudiar homogeneidad.

Contenido y resultados

En el Capítulo 1 sentamos los fundamentos para los siguientes capítulos. Más concretamente, introducimos brevemente la teoría de fibrados principales y conexiones. También definimos la holonomía de una conexión y presentamos los resultamdos más relevantes. Este concepto será central a lo largo de la tesis. A continuación aplicamos esta teoría al marco de las variedades pseudo-Riemannianas y las G-estructuras. Después de enunciar el Teorema de Berger describimos algunas de las estructuras geométricas que trataremos en los capítulos 5, 6 y 7. Finalmente introducimos algunos conceptos básicos sobre espacios homogéneos y definimos la conexión canónica asociada a un espacio reductivo. Este es el ejemplo más representativo de conexión de Ambrose-Singer.

En el Capítulo 2 enunciamos el Teorema de Ambrose-Singer, que constituye el punto de partida de la presente tesis. Primero contextualizamos este resultado relacionándolo con la caracterización de los espacios simétricos dada por Cartan. También enunciamos el Teorema de Kiričenko, el cual extiende el Teorema de Ambrose-Singer a variedades pseudo-Riemannianas equipadas con una estructura geométrica extra. Como el presente autor no ha encontrado una demostración de este resultado en la literatura, se presenta una prueba original. Terminamos este capítulo definiendo la estructura homogénea S asociada a una conexión de Ambrose-Singer. Introducimos el correspondiente modelo infinitesimal, la construcción de Nomizu, y el álgebra de transvecciones, y discutimos algunas de sus propiedades.

En el Capítulo 3 desarrollamos la teoría de conexiones de Ambrose-Singer en variedades localmente homogéneas pseudo-Riemannianas. Probamos que una variedad localmente homogénea pseudo-Riemanniana admite una conexión de Ambrose-Singer si satisface una condición algebraica relacionada con el conjunto de sus campos de Killing locales (Teorema 3.1.9). En analogía con el caso global llamamos a esta condición *reductividad*. Recíprocamente probamos que una variedad pseudo-Riemanniana admitiendo una conexión de Ambrose-Singer es localmente homogénea y reductiva (Teorema 3.1.10). Como es bien sabido, diferentes (pseudo-)grupos de Lie pueden actuar transitivamente sobre la misma variedad. Veremos que la reductividad no es una propiedad de la variedad en sí misma, sino que depende de la acción del (pseudo-)grupo de Lie considerado. Nótese que esto está en concordancia con el espíritu del Programa Erlangen de F. Klein, señalando que las acciones de diferentes (pseudo)-grupos de Lie sobre la misma variedad pueden tener una naturaleza muy distinta. A través de varios ejemplos exploramos los posibles escenarios. Además extenderemos estos resultados al caso en que la variedad esté dotada de una estructura geométrica extra (Teoremas 3.1.16 y 3.1.17).

Siguiendo el trabajo de Kowalski en el caso Riemanniano [40], aparece de manera natural una nueva condición que llamaremos *reductividad fuerte*. Probamos que una variedad localmente homogénea fuertemente reductiva admite una conexión de Ambrose-Singer análoga a la conexión canónica construida por Kowalski (Teorema 3.2.8). Al contario que la condición de redutividad, la fuerte reductividad es una propiedad de la variedad en sí misma y no depende de la acción de ningún (pseudo-)grupo de Lie. Algunas propiedades de las variedades localmente homogéneas pseudo-Riemannianas fuertemente reductivas son estudiadas, y en particular mostramos que este tipo de variedades pueden ser reconstruidas a partir de su curvatura y sus derivadas covariantes hasta orden finito en un punto (Teorema 3.3.2) (recuérdese que esta propiedad la satisfacen todas las variedades localmente homogéneas Riemannianas [49]).

En el Capítulo 4 aprovechamos la naturaleza algebraica de las estructuras homogéneas para obtener resultados de clasificación. Primero esbozamos el procedimiento general para clasificar estructuras homogéneas en presencia y en ausencia de una estructura geométrica extra, y a continuación lo especificamos para las estructuras geométricas y holonomías que aparecen en capítulos siguientes. Definimos las llamadas estructuras homogéneas de tipo lineal que serán el principal objeto de estudio en los Capítulos 5 y 6. Todas las clasificaciones que aparecen en este capítulo fueron obtenidas previamente por diferentes autores con excepción de los casos para-cuaterniónico Kähler, pseudo-hiper-Kähler y para-hiper-Kähler que son originales.

En el Capítulo 5 estudiamos las estructuras homogéneas pseudo-Kähler y para-Kähler de tipo lineal. Probamos que variedades pseudo-Kähler y para-Kähler admitiendo respectivamente estructuras homogéneas pseudo-Kähler y para-Kähler de tipo lineal no degeneradas (véanse Definiciones 4.2.6 y 4.2.9) tienen respectivamente curvatura seccional holomorfa y para-holomorfa constante (Teorema 5.1.2). Así mismo mostramos que, salvo en el caso de métricas definidas, las correspondientes formas espaciales complejas sólo admiten este tipo de estructuras localmente (Teorema 5.3.1). Por otro lado obtenemos la holonomía (Proposiciones 5.2.2 y 5.2.4) y la forma local de la métrica de una variedad pseudo-Kähler o para-Kähler admitiendo respectivamente una estructura homogénea pseudo-Kähler o para-Kähler de tipo lineal degenerada (Proposiciones 5.2.5 y 5.2.6), prestando especial atención a la naturaleza singular de la geometría subyacente. Calculamos los modelos infinitesimales asociados y las álgebras de transvecciones para cada caso, y estudiamos la completitud de los correspondientes modelos homogéneos (Teorema 5.3.2). Finalmente mostramos la relación entre las estructuras degeneradas y cierto tipo de ondas planas homogéneas. Algunos de los resultados contenidos en este capítulo, más concretamente aquellos referidos a las estructuras fuertemente degeneradas, están publicados en [18].

En el Capítulo 6 estudiamos las estructuras homogéneas pseudo-cuaterniónicas y para-cuaterniónicas Kähler de tipo lineal. Por un lado probamos que variedades pseudo-cuaterniónicas y para-cuaterniónicas Kähler admitiendo respectivamente estructuras homogéneas pseudo-cuaterniónicas y para-cuaterniónicas Kähler de tipo lineal no degeneradas (véanse Definiciones 4.2.12 y 4.2.15) tienen respectivamente curvatura seccional quaterniónica y para-cuaterniónica constante (Teorema 6.1.1). Así mismo mostramos que, salvo en el caso de métricas definidas, las correspondientes formas espaciales cuaterniónicas sólo admiten este tipo de estructuras localmente (Teorema 6.2.1). Por otro lado probamos que una variedad pseudo-cuaterniónica o para-cuaterniónicas Kähler admitiendo respectivamente una estructura homogénea pseudo-cuaterniónica o para-cuaterniónica Kähler de tipo lineal degenerada es necesariamente plana (Teorema 6.1.1). Calculamos los modelos infinitesimales y las álgebras de transvecciones asociadas, y estudiamos la completitud de los correspondientes modelos homogéneos (Teorema 6.2.2).

Finalmente, en el Capítulo 7 estudiamos las estructuras homogéneas en el marco de la reducción bajo la acción de un grupo de isometrías. En un primer resultado, H es un subgrupo de normal del grupo de simetrías asociado a una estructura homogénea $\bar{S}$ definida en una variedad gobalmente homogénea. En este caso $\bar{S}$ puede ser reducida a una estructura homogénea en el espacio de órbitas bajo la acción de H (Teorema 7.1.4). En un segundo resultado estudiamos bajo qué condiciones una estructura homogénea $\bar{S}$ definida en el espacio total de un fibrado principal $\pi : (\bar{M}, \bar{g}) \to (M, g)$ reduce a una estructura homogénea en el espacio base (M, g). La respuesta involucra una condición adicional en la llamada conexión mecánica, parecida a la ecuación extra que aparece en el Teorema de Kiričenko. El comportamiento de las clases de estructuras homogéneas pseudo-Riemannianas bajo reducción es analizado (Proposición 7.2.3). Resulta que la geometría de las fibras del fibrado principal está involucrada en la preservación de alguna de ellas (Proposición 7.2.4). Algunos ejemplos clásicos ilustran la teoría. Finalmente el proceso de reducción se aplica a fibraciones de variedades casi-contacto métricas sobre variedades casi-Hermíticas. Cuando la estructura homogénea $\bar{S}$ es cosimpléctica o Sasakiana la estructura homogénea reducida es pseudo-Kähler. Usaremos este resultado para obtener algunas propiedades de estructuras homogéneas cosimplécticas o Sasakianas de tipo lineal (Proposiciones 7.3.8 y 7.3.10). Los contenidos de este capítulo están incluidos en [19].

Conclusiones

Respecto al primer objetivo, hemos sido capaces de extender el Teorema de Ambrose-Singer y la teoría de conexiones de Ambrose-Singer a variedades localmente homogéneas pseudo-Riemannianas. Hemos comprobado que los argumentos usados en el caso Riemanniano no son directamente aplicables al ámbito de métricas con signatura, y una inspección más profunda ha llevado a nuevos conceptos tales como la reductividad y la reductividad fuerte. Estos se revelan como necesarios no sólo para desarrollar la teoría, sino también para extender los resultados del caso Riemanniano. Resulta muy interesante ver cómo la transición del reino Riemanniano al caso de métricas con signatura a menudo aporta perspectiva y revela el caso Riemanniano como una situación muy especial dentro del vasto universo de la geometría pseudo-Riemanniana. En relación con esta idea, es notable lo poco que se sabe sobre variedades homogéneas no reductivas (véase por ejemplo [25]). El Capítulo 3 arroja algo de luz sobre este problema.

Acerca del segundo objetivo, hemos sido capaces de caracterizar la clase de estructuras homogéneas de tipo lineal en los casos pseudo-Kähler, para-Kähler, pseudo-cuaterniónico Kähler y para-cuaterniónico Kähler. Resulta interesante cómo el carácter causal del campo vectorial definiendo estas estructuras separa dos mundos muy diferentes. Por un lado, las estructuras no degeneradas producen resultados análogos al caso Riemanniano, es decir, caracterizan espacios de curvatura seccional holomorfa, para-holomorfa, cuaterniónica y para-cuaterniónica constante. Por otro lado las estructuras degeneradas no poseen análogo Riemanniano, por lo que aparecen nuevos objetos y escenarios. Más concretamente, en los casos pseudo-Kähler y para-Kähler este tipo de estructuras caracterizan variedades cuya geometría puede interpretarse como una generalización (para-)compleja de la geometría de las ondas planas homogéneas. En los casos pseudo-cuaterniónico y para-cuaterniónico, la condición $\widetilde{\nabla}R = 0$ se vuelve demasiado fuerte forzando a la variedad a ser plana. Esto sugiere que la noción de onda plana homogénea no puede generalizarse a las geometrías de tipo cuaterniónico. Otra propiedad resaltable de las estructuras degeneradas y no degeneradas es que de manera natural aparecen problemas de completitud. A parte de que en el caso degenerado la geometría subyacente parece ser singular en un sentido cosmológico, todos los modelos homogéneos asociados a estas estructuras son necesariamente incompletos. El origen de estos problemas de incompletitud aparece difuso en este momento y puede ser un tema de estudio interesante para el futuro.

Finalmente, respecto al tercer objetivo hemos proporcionando un buen esquema de reducción para las estructuras homogéneas. Es notable el papel que juega la geometría del fibrado principal y la condición de Kiričenko impuesta sobre la conexión mecánica. Así mismo merece la pena señalar cómo la geometría de las fibras está involucrada en la preservación de algunas clases de estructuras homogéneas en este proceso de reducción. Como pretendíamos, los resultados obtenidos nos han permitido estudiar nuevos objetos. Más concretamente, hemos podido estudiar algunas propiedades de las estructuras homogéneas cosimplécticas y Sasakianas de tipo lineal a partir del proceso de reducción y los resultados sobre estructuras homogéneas pseudo-Kähler de tipo lineal previamente obtenidos. Muchos problemas sin embargo permanecen abiertos. En primer lugar, puede ser de gran valor tener un proceso inverso a la reducción, es decir, debemos encontrar condiciones que aseguren la posibilidad de construir una estructura homogénea en el espacio total de un fibrado principal a partir de una estructura homogénea en el espacio base. En segundo lugar, el estudio completo de las estructuras homogéneas cosimplécticas y Sasakianas permanece abierto, ya que el proceso de reducción sólo es aplicable a estructuras homogéneas invariantes. En cualquier caso, las propiedades obtenidas en esta parte de la tesis adelantan gran parte del trabajo y proporcionan una ventaja significativa.

Los resultados derivados del trabajo realizado en esta tesis doctoral se encuentran en [18], [19], [20], [27], [41], y [42].

Chapter 1

Preliminaries

In this chapter we settle the foundations for subsequent chapters. We will recall some basic definitions and results which will be used throughout the rest of the manuscript in order to be as self-contained as possible.

1.1 Principal bundles and connections

The base text for this section is [38], where all the proofs not appearing here can be found. Unless otherwise stated all objects are assumed to be $\mathcal{C}^\infty$.

1.1.1 Principal bundles

Definition 1.1.1 *Let P and M be manifolds, and let G be a Lie group. A principal bundle $P(M,G)$ is a surjective submersion $\pi : P \to M$, such that G acts freely and transitively on the right on the fibers of π.*

The manifolds P and M are called the *total space* and the *base space* respectively, and G is called the *structure group*. The action of $a \in G$ on $u \in P$ will be denoted by $R_a(u)$ or simply $u \cdot a$. For every $p \in M$ there is a neighborhood U which is the domain of a local section $\sigma : U \to \pi^{-1}(U)$. Then, $\phi : U \times G \to \pi^{-1}(U)$, $\phi(q,a) = R_a(\sigma(q))$ is a diffeomorphism such that $\pi \circ \phi(q,a) = q$ and $\phi(R_a(u)) = (q, ba)$, where $\phi(u) = (q,b)$. The most important example of principal bundle for our purposes is the so called *bundle of references* or *bundle of frames*.

Example 1.1.2 *Let M be a manifold of dimension m. We consider the set*

$$\mathcal{L}(M) = \{u = (p; u_1, \ldots, u_n) / \, p \in M, \, (u_1, \ldots, u_m) \text{ is a basis of } T_pM\},$$

which is easily seen to have a structure of differentiable manifold. The natural projection $\pi : \mathcal{L}(M) \to M$ defines a principal bundle structure with structure group $GL(m, \mathbb{R})$. The action of a matrix $a = (a_{ij}) \in GL(m,\mathbb{R})$ on $u = (p; u_1, \ldots, u_m)$ is defined by $R_a(u) = (p; \widetilde{u}_1, \ldots, \widetilde{u}_m)$ with $\widetilde{u}_j = \sum_i a_{ij} u_i$. For the sake of simplicity we will often omit the point p when describing a reference and we will only write $u = (u_1, \ldots, u_m)$. In addition, it will be very useful to interpret references as a linear isomorphisms

$$\begin{array}{rccl} u: & \mathbb{R}^m & \to & T_pM \\ & \eta & \mapsto & \sum_{i=1}^m \eta_i u_i. \end{array}$$

Let $P(M,G)$ be a principal bundle with structure group G, and let F be a manifold on which G acts on the left. G acts on $P \times F$ on the right as $(u,f) \cdot a = (u \cdot a, a^{-1} \cdot f)$. The quotient $E = P \times_G F = (P \times F)/G$ together with the projection $\pi_E : E \to M$, $\pi_E([u,v]_G) = \pi(u)$ is a bundle called the *associated bundle to $P(M,G)$ with fiber F*. When $F = V$ is a vector space and G acts linearly on V, the associated bundle $E = P \times_G V$ is a vector bundle.

Example 1.1.3 *Let $V=\mathbb{R}^m$ be endowed with the standard left action of $GL(m,\mathbb{R})$. As a straightforward computation shows, the associated vector bundle $E=\mathcal{L}(M)\times_{GL(m,\mathbb{R})}\mathbb{R}^m$ is isomorphic to the tangent bundle TM of M. In the same way, the vector bundle $\mathcal{T}^r_s(M)$ of tensor fields of type (r,s) on M can be modelled as the vector bundle associated to $\mathcal{L}(M)$ with fiber $V=(\otimes^s(\mathbb{R}^m)^*)\otimes(\otimes^r\mathbb{R}^m)$.*

There is a one to one correspondence between equivariant maps $f:P\to V$ (that is, $f(R_a(u))=a^{-1}\cdot f(u)$) and sections $\sigma:M\to E$. We associate to every equivariant map f the section $\sigma(p)=[u,f(u)]_G$, where u is any element in $\pi^{-1}(p)$. Conversely, we associate to every section σ the equivariant map $f(u)=\eta$, where $[u,\eta]_G=\sigma(\pi_E(u))$.

A *homomorphism* between two principal bundles $P'(M',G')$ and $P(M,G)$ is a map $\Psi:P'\to P$ together with a homomorphism of Lie groups $\gamma:G'\to G$ such that $\Psi(R_{a'}(u'))=R_{\gamma(a')}(\Psi(u'))$. Each homomorphism Ψ of principal bundles induces a map $\psi:M'\to M$ with $\pi\circ\Psi=\psi\circ\pi'$.

Definition 1.1.4 *We say that $P'(M',G')$ is a subbundle of $P(M,G)$ if there is a homomorphism $i:P'(M',G')\to P(M,G)$ such that $i:P\to P'$ is an embedding and $\gamma:G'\to G$ is a monomorphism. If moreover $M=M'$, and the map induced in the base manifolds is the identity transformation, then $P'(M',G')$ is called a reduction of $P(M,G)$ to structure group G'.*

For us, the most important examples of reduction are the so called *G-structures*, that is, reductions of $\mathcal{L}(M)$ to a subgroup $G\subset GL(m,\mathbb{R})$. The reason is that under suitable conditions a G-structure will determine a geometric structure on M and viceversa.

Example 1.1.5 *Let (M,g) be a pseudo-Riemannian manifold with signature (r,s). We consider the set*

$$\mathcal{O}(M)=\left\{u\in\mathcal{L}(M)/\ u \text{ is an orthonormal basis of } (T_{\pi(u)},g_{\pi(u)})\right\}.$$

The natural inclusions $i:\mathcal{O}(M)\hookrightarrow\mathcal{L}(M)$ and $i:O(r,s)\hookrightarrow GL(m,\mathbb{R})$ determine a reduction of $\mathcal{L}(M)$ to structure group $O(r,s)$. Conversely, every $O(r,s)$-reduction P of $\mathcal{L}(M)$ determines a pseudo-Riemannian metric g on M so that P is the bundle of orthonormal frames of g.

Remark 1.1.6 *Let $P'(M,G')$ be a reduction of $P(M,G)$. Let V be a vector space on which G acts on the left (hence so does G' by restriction). It is easy to see the associated bundles to $P'(M,G')$ and $P(M,G)$ with fiber V are isomorphic, that is*

$$P\times_G V=P'\times_{G'}V.$$

This implies that defining a section of these associated bundles is equivalent to give a G-equivariant map $P\to V$ or a G'-equivariant map $P'\to V$.

1.1.2 Connections on principal bundles

Let $P(M,G)$ be a principal bundle. For every $u\in P$ we define the vertical subspace $V_uP\subset T_uP$ at u as the tangent space to the fiber $\pi^{-1}(\pi(u))$ at u. Let $\mathfrak{g}$ be the Lie algebra of G and let $A\in\mathfrak{g}$. We define the *fundamental vector field* A^* associated to A by

$$A_u=\left.\frac{d}{dt}\right|_{t=0}R_{exp(tA)}(u),\qquad u\in P.$$

It is easy to see that $A\mapsto A^*_u$ is an isomorphism between $\mathfrak{g}$ and V_uP. Moreover, by the property $(R_b)_*(A^*)=(\mathrm{Ad}(b^{-1})A)^*$, it determines a Lie algebra homomorphism between $\mathfrak{g}$ and $\mathfrak{X}(P)$.

Definition 1.1.7 *A connection Γ on a principal bundle $P(M,G)$ is a G-equivariant distribution HP complementary to the vertical distribution VP, that is, for every $u \in P$ we can write $T_uP = H_uP \oplus V_uP$ smoothly with respect to u and such that*

$$(R_a)_*(H_uP) = H_{R_a(u)}P, \qquad u \in P, a \in G.$$

HP is called the *horizontal distribution.* Let $X_u \in T_uP$, we can write $X_u = X_u^h + X_u^h$, where X_u^h and X_u^v denotes the horizontal and vertical part of X_u with respect to Γ respectively. We define the 1-form ω on P with values in $\mathfrak{g}$ given by $\omega_u(X_u) = A$, where A is the unique element of $\mathfrak{g}$ with $A_u^* = X_u^v$. The form ω is called the *connection form* of Γ. As a simple inspection shows there is a one to one correspondence between connections Γ on $P(M,G)$ and 1-forms ω on P with values in $\mathfrak{g}$ satisfying

1. $\omega(X) = 0$ if and only if X is horizontal.
2. $\omega(A^*) = A$ for every $A \in \mathfrak{g}$.
3. ω is G-equivariant, i.e., $(R_a)^* \omega = \mathrm{Ad}\left(a^{-1}\right) \omega$ for every $a \in G$.

Let $X_p \in T_pM$ and let $u \in \pi^{-1}(p)$. We define the *horizontal lift* of X_p to u as the unique vector $X_u^H \in H_uP$ such that $\pi_*(X_u^H) = X_p$. We thus have that for every vector field $X \in \mathfrak{X}(M)$ there is a unique horizontal vector field X^H such that it is G-equivariant and $\pi_*(X^H) = X$. In addition one has $[X^H, Y^H]^h = [X,Y]^H$. A $\mathcal{C}^1$ curve on P is called *horizontal* if its tangent vectors are horizontal at every point. This way, for every $\mathcal{C}^1$ curve τ_t on M and every $u_0 \in P$ there is a unique horizontal curve $\bar{\tau}_t$ on P such that $\bar{\tau}_0 = u_0$ and $\pi(\bar{\tau}_t) = \tau_t$. The curve $\bar{\tau}$ is called the *horizontal lift* of τ to u_0 with respect to the connection Γ. Let τ_t, $0 \le t \le 1$, be a $\mathcal{C}^1$ curve on M, and let $\bar{\tau}_t$ be its horizontal lift to a point $u_0 \in \pi^{-1}(\tau_0)$. The end point $u_1 = \bar{\tau}_1$ will be a point in the fiber $\pi^{-1}(\tau_1)$. This defines a map (which we will also denote by τ)

$$\begin{array}{rccc} \tau: & \pi^{-1}(\tau_0) & \to & \pi^{-1}(\tau_1) \\ & u_0 & \mapsto & u_1 \end{array}$$

called the *parallel transport* along the curve τ with respect to the connection Γ. It is immediate that the parallel transport commutes with the action of G, that is, $R_a \circ \tau = \tau \circ R_a$, and that it is independent of the parametrization of τ. In addition, the parallel transport along the inverse curve of τ_t is the map τ^{-1} (in particular τ is an isomorphism) and the parallel transport along the composition of two curves is the composition of the corresponding maps.

Definition 1.1.8 *Let Γ be a connection on $P(M,G)$ and ω its connection form. The 2-form Ω with values in $\mathfrak{g}$ defined by*

$$\Omega(X,Y) = d\omega(X^h, Y^h)$$

is called the curvature form of Γ. Ω is horizontal and satisfies $R_a^*\Omega = \mathrm{Ad}(a^{-1})\Omega$.

Theorem 1.1.9 (Structure equation) *Let Ω be the curvature of a connection ω. Then*

$$\Omega(X,Y) = d\omega(X,Y) + [\omega(X), \omega(Y)],$$

where the brackets are the Lie algebra brackets of $\mathfrak{g}$.

Note that if X, Y are horizontal then $\Omega(X,Y) = -\omega([X,Y])$, so that the curvature form gives the vertical part of the bracket of two horizontal vector fields.

Let $\Psi : P'(M', G') \to P(M,G)$ be a homomorphism of principal bundles with homomorphism of Lie groups $\gamma : G' \to G$, and with $\psi : M' \to M$ a diffeomorphism. Let Γ' be a connection on $P'(M', G')$ with connection form and curvature form ω' and Ω'

respectively. Then there is a unique connection Γ on $P(M,G)$ such that Ψ takes the horizontal subspaces of P' with respect to Γ' to the horizontal subspaces of P with respect to Γ. Moreover, let ω and Ω be the connection form and curvature form of Γ, then $\Psi^*\omega = \gamma \circ \omega'$ and $\Psi^*\Omega = \gamma \circ \Omega'$. In these conditions we say that Ψ takes Γ' to Γ. In the particular case when $P'(M',G')$ is a reduction of $P(M,G)$ we say that Γ is *reducible* to $P'(M',G')$. On the other hand, if an automorphism Ψ of a principal bundle $P(M,G)$ takes a connection Γ to itself we say that Γ is *invariant* by Ψ.

We now relate the notion of a connection on a principal bundle with the well known notion of covariant derivative on a vector bundle. Let $P(M,G)$ be a principal bundle and Γ a connection on $P(M,G)$. Let E be an associated vector bundle with fiber a vector space V. We can endow E with a notion of parallel transport inherited form Γ in the following way. For $w \in E$ we define the vertical subspace $V_wE \subset T_wE$ as the tangent space to the fiber $\pi_E^{-1}(\pi_E(w))$ at w. In order to define a horizontal subspace we consider the natural projection $P \times V \to E = P \times_G V$, $(u,\eta) \mapsto [u,\eta]$, and we take a point (v,ξ) such that $w = [v,\xi]$. Fixing ξ we consider the map

$$\begin{array}{ccc} P & \to & E \\ u & \mapsto & [u,\xi]. \end{array}$$

Then, H_wE is defined as the image of H_vP by the differential of this map (which is independent of the choice of (v,ξ)), and it is easy to see that $T_wE = V_wE \oplus H_wE$. This way, a curve in E is said to be *horizontal* if its tangent vectors are horizontal at every point. As expected, given a curve γ_t in M and a point w_0 in the fiber of γ_0 there is a unique *horizontal lift* $\bar{\gamma}_t$ in E starting at w_0. Therefore, the *parallel transport* along a curve γ_t, $0 \leq t \leq 1$, is defined analogously to the case of principal bundles, resulting in this case a linear isomorphism $\gamma : \pi_E^{-1}(\gamma_0) \to \pi_E^{-1}(\gamma_1)$. A section $\phi : M \to E$ will be called *parallel* whenever $\phi_*(T_pM) \subset H_{\phi(p)}E$ for every $p \in M$, or equivalently, if the parallel transport along any curve γ_t takes $\phi(\gamma_0)$ to $\phi(\gamma_1)$. We shall denote by $\gamma_{t_1}^{t_2}$ the parallel transport along γ between γ_{t_1} and γ_{t_2}.

Definition 1.1.10 *Let ϕ be a section of E and γ_t, $-\epsilon \leq t \leq \epsilon$, a curve in M. The covariant derivative of ϕ along γ at γ_0 is given by*

$$\nabla_{\dot{\gamma}_0}\phi = \lim_{t\to 0} \frac{1}{t}[\gamma_t^0(\phi(\gamma_t)) - \phi(\gamma_0)] \in \pi_E^{-1}(\gamma_0).$$

The covariant derivative of ϕ at $p \in M$ in the direction of a tangent vector $X_p \in T_pM$ is just defined as the covariant derivative of ϕ along a curve γ_t at γ_0, where $\gamma_0 = p$ and $\dot{\gamma}_0 = X_p$. In addition, the covariant derivative of ϕ in the direction of a vector field X is the section $\nabla_X\phi : M \to E$, $p \mapsto \nabla_{X_p}\phi$. Recall on the other hand, that sections of E can be interpreted as G-equivariant maps $\phi : P \to V$. It is easy to see that the G-equivariant map corresponding to the section $\nabla_X\phi$ is $X^H\phi : P \to V$, that is, the horizontal lift of X differentiating the function $\phi : P \to V$.

We now focus on the so called *linear connections*, which are connections defined on the principal bundle $\mathcal{L}(M)$. Recall that the bundle $\mathcal{T}_s^r(M)$ of tensors of type (r,s) can be seen as an associated bundle to $\mathcal{L}(M)$. This way one can recover the usual covariant derivative of a tensor field. Hereafter we will interchangeably interpret a reference $u \in \mathcal{L}(M)$ as a basis of $T_{\pi(u)}M$ or as a linear isomorphism $u : \mathbb{R}^m \to T_{\pi(u)}M$, and we will not distinguish between the covariant derivative ∇ and the linear connection Γ.

Definition 1.1.11 *Let ∇ be a linear connection on M, we define the curvature tensor field of ∇ as the $(1,3)$ tensor field*

$$R(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z),$$

and the torsion field of ∇ *as the* $(1,2)$ *tensor field*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

As tensor fields, R and T are associated to $GL(m,\mathbb{R})$-equivariant functions from $\mathcal{L}(M)$ to the corresponding space of tensors. We now see which these functions are.

We define the *contact form* θ of $\mathcal{L}(M)$ as the $\mathbb{R}^m$ valued 1-form given by $\theta(X_u) = u^{-1}(\pi_*(X_u))$, for $X_u \in T_u\mathcal{L}(M)$. One can check that θ satisfies $R_a^*\theta = a^{-1}\cdot\theta$ for $a \in GL(m,\mathbb{R})$. Let $u \in \mathcal{L}(M)$, to every $\eta \in \mathbb{R}^m$ we can associate in a unique way a horizontal vector $B(\eta)_u \in T_u\mathcal{L}(M)$ such that $\pi_*(B(\eta)) = u(\eta)$. The vector field $B(\eta)$ is called the *standard vector field* associated to η. It is obvious that standard vector fields depend on the chosen connection. They are nowhere vanishing for $\eta \neq 0$ and satisfy $\theta(B(\eta)) = \eta$, and $(R_a)_*(B(\eta)) = B(a^{-1}\eta)$ for $a \in GL(m,\mathbb{R})$. In addition, for $A \in \mathfrak{gl}(m,\mathbb{R})$ and $\eta \in \mathbb{R}^m$ one has $[A^*, B(\eta)] = B(A\eta)$. The *torsion form* Θ of a linear connection Γ is defined as $\Theta(X,Y) = d\theta(X^h, Y^h)$. In particular $R_a^*\Theta = a^{-1}\cdot\Theta$ for $a \in GL(m,\mathbb{R})$, and it satisfies the structure equation

$$\Theta = d\theta + \omega \wedge \theta.$$

The proof of the following proposition follows immediately.

Proposition 1.1.12 *Let* ∇ *be a linear connection on* M*.*

1. *The equivariant function associated with the torsion tensor field of* ∇ *is*

$$\begin{array}{rrcl} t: & \mathcal{L}(M) & \to & (\mathbb{R}^m)^* \otimes (\mathbb{R}^m)^* \otimes \mathbb{R}^m \\ & u & \mapsto & T(u)(\eta,\xi) = \Theta_u(B(\eta)_u, B(\xi)_u). \end{array}$$

2. *The equivariant function associated with the curvature vector field of* ∇ *is*

$$\begin{array}{rrcl} r: & \mathcal{L}(M) & \to & (\otimes^3(\mathbb{R}^m)^*) \otimes \mathbb{R}^m \\ & u & \mapsto & R(u)(\eta,\xi,\zeta) = \Omega_u(B(\eta)_u, B(\xi)_u)\zeta. \end{array}$$

Definition 1.1.13 *Let* M *and* M' *be manifolds with linear connections* Γ *and* Γ'*. We say that* $f : M \to M'$ *is an affine map if* $f_* : TM \to TM'$ *takes horizontal curves with respect to* Γ *to horizontal curves with respect to* Γ'*. If* $f : M \to M$ *is moreover a diffeomorphism, then it is called an affine transformation. An affine map satisfies in particular* $f_*(\nabla_X Y) = \nabla'_{f_*X} f_*Y$*,* $f_*(R(X,Y)Z) = R'(f_*X, f_*Y)f_*Z$*, and* $f_*(T(X,Y)) = T'(f_*X, f_*Y)$*.*

Any transformation $f : M \to M$ induces a transformation of principal bundles $\widetilde{f} : \mathcal{L}(M) \to \mathcal{L}(M)$ given by $\widetilde{f}(u) = (f_*(u_1), \ldots, f_*(u_m))$. In particular $\widetilde{f}$ preserves fundamental vector fields and the contact form θ. If f is moreover an affine transformation, then $\widetilde{f}^*\omega = \omega$.

1.1.3 Holonomy

In this section we define the concept of holonomy of a connection, which will be central throughout the thesis. Its importance resides in the fact that it contains great part of the geometric information of the principal bundle and the connection. Although for the following definitions and results one can work with curves of class $\mathcal{C}^k$, $0 \leq k \leq \infty$ (see [38, Ch. II, §7]), for simplicity will only consider curves of class $\mathcal{C}^\infty$.

Let $P(M,G)$ be a principal bundle endowed with a connection Γ. For every $p \in M$ we denote by $C(p)$ the space of loops based at p. Let $\tau \in C(p)$, we consider the parallel transport along τ with respect to Γ, which will be an automorphism

$$\tau : \pi^{-1}(p) \to \pi^{-1}(p).$$

The set of all parallel transports along loops based at p forms a group.

Definition 1.1.14 *The group*

$$Hol(p) = \{\tau : \pi^{-1}(p) \to \pi^{-1}(p) / \, \tau \in C(p)\}$$

is called the holonomy group of Γ *at* p.

Let $C^0(p)$ be the subset of $C(p)$ consisting of contractible loops based at p. The subgroup

$$Hol^0(p) = \{\tau : \pi^{-1}(p) \to \pi^{-1}(p) / \, \tau \in C^0(p)\}$$

is called the *restricted holonomy group* of Γ based p. It will be very convenient to see these groups as subgroups of the structure group G in the following way. Let $\tau \in C(p)$ and $u_0 \in \pi^{-1}(p)$ fixed, then $\tau(u_0) \in \pi^{-1}(p)$. Hence there is an element $a \in G$ such that $\tau(u_0) = R_a(u_0)$. We can thus identify the automorphism τ with the element $a \in G$, so that $Hol(p)$ is seen as a subgroup $Hol(u_0)$ of G, called the holonomy group of Γ with base point u_0. Considering contractible loops one defines the restricted Holonomy subgroup $Hol^0(u_0) \subset Hol(u_0) \subset G$. A third way to define the holonomy group of Γ is considering the equivalence relation $u \sim v$ if and only if u and v can be joined by an horizontal curve. Then it is immediate that

$$Hol(u_0) = \{a \in G / \, u_0 \sim R_a(u_0)\}.$$

It is easy to see that for $u, v \in P$, if $\pi(u)$ and $\pi(v)$ can be joined by a curve, then there is an element $a \in G$ with $u \sim R_a(v)$, so that $Hol(u)$ and $Hol(v)$ are conjugated. The same holds for the restricted groups.

The following Theorem is one of the most important results in Holonomy Theory. The proof can be found once again in [38].

Theorem 1.1.15 *Let* $P(M, G)$ *be a principal bundle endowed with a connection* Γ, *where* M *a connected and paracompact. Let* $\mathrm{Hol}(u)$ *and* $\mathrm{Hol}^0(u)$ *be the holonomy group and the restricted holonomy group of* Γ *based at* $u \in P$. *Then*

1. $\mathrm{Hol}^0(u)$ *is a connected Lie subgroup of* G.
2. $\mathrm{Hol}^0(u)$ *is a normal subgroup of* $\mathrm{Hol}(u)$ *and* $\mathrm{Hol}(u)/\mathrm{Hol}^0(u)$ *is countable.*

These imply that $\mathrm{Hol}(u)$ *is a Lie subgroup of* G, *whose connected component containing the identity is* $\mathrm{Hol}^0(u)$.

Concerning the behavior of the holonomy groups under homomorphisms of principal bundles we have the following result.

Proposition 1.1.16 *Let* $\Psi : P'(M', G') \to P(M, G)$ *be a homomorphism of principal bundles. Let* $\gamma : G' \to G$ *and* $\psi : M' \to M$ *be the corresponding maps.*

1. *If* ψ *is a diffeomorphism and* $\Psi(u') = u$, *then* γ *takes* $\mathrm{Hol}(u')$ *to* $\mathrm{Hol}(u)$ *and* $\mathrm{Hol}^0(u')$ *to* $\mathrm{Hol}^0(u)$.
2. *If* γ *is an isomorphism and* $\Psi(u') = u$, *then* γ *takes* $\mathrm{Hol}(u')$ *to* $\mathrm{Hol}(u)$ *and* $\mathrm{Hol}^0(u')$ *to* $\mathrm{Hol}^0(u)$.

Let $u \in P$ be fixed, we consider the set

$$\mathcal{P}(u) = \{v \in P / \, v \sim u\}.$$

It is easy to see that $\mathcal{P}(u)$ is principal bundle called the *holonomy bundle* of Γ based at u. It is obvious that $\mathcal{P}(u) = \mathcal{P}(v)$ if and only if $u \sim v$, and if $u \nsim v$ then $\mathcal{P}(u) \cap \mathcal{P}(v) = \emptyset$. Recall that the action of G takes horizontal curves to horizontal curves, hence for every $a \in G$ we have that $R_a : \mathcal{P}(u) \to \mathcal{P}(R_a(u))$ is an isomorphism of principal bundles with the corresponding isomorphism of Lie groups $\mathrm{Ad}(a^{-1}) : Hol(u) \to Hol(R_a(u))$. Since for every $u, v \in P$ there exist an element $a \in G$ such that $u \sim R_a(v)$, the holonomy bundles $\mathcal{P}(u)$ and $\mathcal{P}(v)$ are isomorphic for every $u, v \in P$.

Theorem 1.1.17 (Reduction Theorem) *Let $P(M,G)$ be a principal bundle with a connection Γ, and let u_0 be a fixed point of P. Then $\mathcal{P}(u_0)$ is a reduction of $P(M,G)$ to group $Hol(u_0)$. Moreover, the connection Γ is reducible to $\mathcal{P}(u_0)$.*

Theorem 1.1.18 (Holonomy Theorem) *Let $P(M,G)$ be a principal bundle with a connection Γ. Let Ω be the curvature form of Γ and $\mathcal{P}(u)$ its holonomy bundle with base point $u \in P$. Then the Lie algebra of $Hol(u)$ is the subalgebra $\mathfrak{hol}(u) \subset \mathfrak{g}$ spanned by all the elements of the form $\Omega_v(X,Y)$, where $v \in \mathcal{P}(u)$ and $X, Y \in H_vP$.*

1.2 Pseudo-Riemannian connections, G-structures, and Berger's Theorem

1.2.1 Pseudo-Riemannian connections and G-structures

Let (M,g) be a pseudo-Riemannian manifold with signature (r,s), and $\mathcal{O}(M)$ be the corresponding bundle of orthonormal frames.

Definition 1.2.1 *A linear connection is called metric if it is reducible to $\mathcal{O}(M)$.*

Proposition 1.2.2 *A linear connection ∇ is metric if and only if $\nabla g = 0$.*

This proposition is a special case of a more general result given at the end of the section. The following Theorem is a well known result

Theorem 1.2.3 *There is a unique metric and torsionless linear connection on (M,g) called the Levi-Civita connection of g. It is obtained by*

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ & + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X). \end{aligned}$$

Unless otherwise specified, hereafter ∇ will denote the Levi-Civita connection of (M,g). Let R be the curvature tensor field of ∇, since ∇ is uniquely determined by g, we will refer to R as the curvature tensor field of g. We will also interpret R both as a $(1,3)$-tensor field and as a $(0,4)$-tensor field by means of the formula $R_{XYZW} = g(R_{XY}Z, W)$. It satisfies the following symmetries:

$$\begin{aligned} & R_{XYZW} = -R_{YXZW}, \\ & R_{XYZW} = R_{ZWXY}, \\ & \underset{XYZ}{\mathfrak{S}} R_{XYZW} = 0 \quad \text{(first Bianchi identity)}, \\ & \underset{XYZ}{\mathfrak{S}} (\nabla_X R)_{YZWV} \quad \text{(second Bianchi identity)}. \end{aligned}$$

We define the *Ricci tensor field* and the *scalar curvature* of g as

$$Ric_{XY} = \sum_{i=1}^{m} R_{e_i X e_i Y}, \qquad \mathrm{s} = \sum_{i=1}^{m} Ric_{e_i e_i},$$

where $\{e_i\}$ is any orthonormal basis.

Definition 1.2.4 *Let (M,g) and (M',g') be pseudo-Riemannian manifolds. A map $f : M \to M'$ is called an isometry if it is a diffeomorphism and the differential $f_{*,p} : (T_pM, g_p) \to (T_{f(p)}M', g'_{f(p)})$ is a linear isometry at every point $p \in M$.*

We will say that (M, g) and (M', g') are *isometric* if there is an isometry between them. We will say that (M, g) and (M', g') are *locally isometric* if for every pair of points $p \in M$ and $q \in M'$ there are neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of p and q respectively, and an isometry $f : \mathcal{U} \to \mathcal{V}$ with $f(p) = q$.

Proposition 1.2.5 *A diffeomorphism $f : M \to M$ is an isometry if and only if the induced map $\widetilde{f} : \mathcal{L}(M) \to \mathcal{L}(M)$ restricts to a map $\widetilde{f} : \mathcal{O}(M) \to \mathcal{O}(M)$.*

The proof of the previous Proposition is evident since every linear isometry sends orthonormal basis to orthonormal basis. The intimate relation between the Levi-Civita connection and the metric can be notice with this Proposition.

Proposition 1.2.6 *Let $f : (M, g) \to (M', g')$ be a diffeomorphism.*

1. *If f is an isometry then it is an affine map with respect to the Levi-Civita connections of g and g'.*

2. *Let $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ be metric connections on (M, g) and (M', g'). If f is an affine map with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ and $f_{*,p}$ is a linear isometry for some point $p \in M$, then f is an isometry.*

The following result exhibits the rigidity of isometries.

Proposition 1.2.7 *Let $f, h : (M, g) \to (M', g')$ be two isometries between connected manifolds. If there is a point $p \in M$ such that $f(p) = h(p)$ and $f_{*,p} = h_{*,p}$, then $f = h$.*

A vector field $X \in \mathfrak{X}(M)$ is called an *infinitesimal isometry* or a *Killing vector field* if its one parameter group of local transformations consists of local isometries. Analogously, X is called an *infinitesimal affine transformation* if its one parameter group consists of local affine maps.

Proposition 1.2.8 *Let $X \in \mathfrak{X}(M)$. The following are equivalent*

1. *X is a Killing vector field.*

2. *$\mathcal{L}_X g = 0$.*

3. *The horizontal lift X^H of X with respect to the Levi-Civita connection is tangent to $\mathcal{O}(M)$.*

The set of all isometries $f : (M, g) \to (M, g)$ has a group structure with the usual composition of maps. This group is called the *isometry group* of (M, g), and will be denoted by *Isom*(M, g) or simply *Isom*(M). One of the main results concerning the isometry group is the following.

Theorem 1.2.9 *The isometry group of a pseudo-Riemannian manifold (M, g) with a finite number of connected components is a Lie group with the compact-open topology.*

We have seen the relation between pseudo-Riemannian metrics and reductions of $\mathcal{L}(M)$ to structure group $O(r, s)$, and how a special connection can be defined in that reduction. This idea can be generalized to other geometries related to some G-structures. We will see under which conditions a connection in $\mathcal{L}(M)$ can be reduced to a G-structure, and what this imposes on the holonomy group.

Let $K_0 \in (\otimes^r (\mathbb{R}^m)^*) \otimes \left(\otimes^l \mathbb{R}^m\right)$ be a tensor of type (r, l). Let $H \in GL(m, \mathbb{R})$ be the stabilizer of K_0 under the action of $GL(m, \mathbb{R})$, that is

$$H = \{a \in GL(m, \mathbb{R}) /\, a \cdot K_0 = K_0\}.$$

Suppose that there is tensor field K of type (r,l) on M, such that the associated equivariant map

$$k : \mathcal{L}(M) \to (\otimes^r(\mathbb{R}^m)^*) \otimes \left(\otimes^l \mathbb{R}^m\right)$$

takes values in the $GL(m,\mathbb{R})$-orbit of K_0. Then it is easy to see that the set $Q = k^{-1}(K_0)$ defines a reduction of $\mathcal{L}(M)$ to structure group H, that is, an H-structure. It is worth noting that Q is the set of references with respect to which K is expressed as K_0. In addition, if K_0 and K_0' are in the same $GL(m,\mathbb{R})$-orbit, then their stabilizers H and H' are conjugated, and the H-structure defined by K_0 and the H'-structure defined by K_0' are isomorphic. Conversely, let Q be an H-structure such that H is the stabilizer inside $GL(m,\mathbb{R})$ of a tensor $K_0 \in (\otimes^r(\mathbb{R}^m)^*) \otimes \left(\otimes^l \mathbb{R}^m\right)$. We define the following H-equivariant map

$$\begin{array}{rccl} k: & Q & \to & (\otimes^r(\mathbb{R}^m)^*) \otimes \left(\otimes^l \mathbb{R}^m\right) \\ & u & \mapsto & K_0. \end{array}$$

This map can be extended to $\mathcal{L}(M)$ by $GL(m,\mathbb{R})$-equivariance, defining this way a tensor field on M. We have thus proved the following

Proposition 1.2.10 *Let H be the stabilizer inside $GL(m,\mathbb{R})$ of a tensor K_0. There is a one to one correspondence between H-structures and tensor fields K on M such that k takes values in the $GL(m,\mathbb{R})$-orbit of K_0.*

Moreover (see for instance [53, Lemma 1.3])

Proposition 1.2.11 *Let Q be an H-structure with H the stabilizer inside $GL(m,\mathbb{R})$ of a tensor K_0. Let K be the associated tensor field on M. A linear connection $\widetilde{\nabla}$ reduces to Q if and only if $\widetilde{\nabla} K = 0$.*

We will say that a G-structure $P(M,G)$ is *integrable* if there is a linear connection with vanishing torsion which reduces to $P(M,G)$.

1.2.2 Berger's Theorem

We begin this section showing the relation between Proposition 1.2.11 and the holonomy of a pseudo-Riemannian manifold. For a proof of the following Proposition see [11, p. 282].

Proposition 1.2.12 (Equivalence Principle) *Let (M,g) be a pseudo-Riemannian manifold. Let H be the stabilizer inside $O(r,s)$ of a tensor K_0. The following statements are equivalent:*

1. *There is a tensor field K on M whose equivariant map k takes values in the $O(r,s)$-orbit of K_0 and such that $\nabla K = 0$.*
2. *There is a reduction $Q(M,H)$ of $\mathcal{O}(M)$ which is integrable.*
3. *$Hol(u_0) \subset H$ for $u_0 \in \mathcal{O}(M)$.*

The celebrated Theorem by Berger [9, 10] provides a list (which was refined later by several authors) of the possible groups appearing as the holonomy group of an irreducible non-locally symmetric pseudo-Riemannian manifold. This result in conjunction with the decomposition Theorems by de Rham and Wu (see [24] and [65]) gives a classification of the possible geometric structures admitted by a pseudo-Riemannian manifold. Before stating the Theorem we need some definitions and facts.

Definition 1.2.13 *Let G be a group and V a vector space.*

1. *A representation (ρ, V) of G on V is said irreducible if there is no proper invariant subspace of V.*

2. *Let V be endowed with an inner product $\langle\, ,\, \rangle$. A representation (ρ, V) is said orthogonal with respect to $\langle\, ,\, \rangle$ if every automorphism $\rho(g) : V \to V$, $g \in G$, is an isometry with respect to $\langle\, ,\, \rangle$. In that case, (ρ, V) is said indecomposable if $\langle\, ,\, \rangle$ is degenerate on every proper invariant subspace of V. This concept is also known in the literature as weakly irreducible.*

It is evident that an orthogonal irreducible representation is always indecomposable. The converse holds only for definite inner products.

Let (M, g) be a pseudo-Riemannian manifold of signature (r, s), and let $p \in M$. The parallel transport along a loop based at p with respect to the Levi-Civita connection gives a transformation of $O(T_pM)$, which can be identified with $O(r, s)$ by fixing an orthonormal basis. The holonomy group $Hol(p)$ of the Levi-Civita connection is thus seen as a subgroup of $O(r, s)$ which acts orthogonally on (T_pM, g_p). We refer to this representation as the *holonomy representation*. When the holonomy representation is irreducible we say that (M, g) is *irreducible*, and when the holonomy representation is indecomposable we say that (M, g) is *indecomposable*. Recall that if (M_1, g_1) and (M_2, g_2) are pseudo-Riemannian manifolds, the product $M_1 \times M_2$ with the metric $g = g_1 + g_2$ is a pseudo-Riemannian metric whose holonomy group is $Hol_g(p_1, p_2) = Hol_{g_1}(p_1) \times Hol_{g_2}(p_2)$ acting on $T_{p_1}M_1 \oplus T_{p_2}M_2$ as the product representation. The converse result is stated in the following Theorem.

Theorem 1.2.14 (de Rham, Wu) *Let (M, g) be a pseudo-Riemannian manifold and $p \in M$. Then there exists an orthogonal decomposition of T_pM into invariant subspaces*

$$T_pM = E_0 \oplus E_1 \oplus \ldots \oplus E_l,$$

such that $Hol(p)$ acts trivially on E_0 and indecomposably on $E_1, \ldots, E_l$, and

$$Hol(p) = \{id\} \times Hol(p)|_{E_1} \times \ldots \times Hol(p)|_{E_l}.$$

Furthermore, if (M, g) is simply-connected and complete, then it is isometric to the product

$$(N_0, g_0) \times (N_1, g_1) \times \ldots \times (N_l, g_l),$$

where (N_0, g_0) is flat, $T_pN_i = E_i$, $g_i = g|_{E_i}$, and $Hol_{g_i}(p) = Hol(p)|_{E_i}$ for $i = 1, \ldots, l$. If (M, g) is not simply-connected or complete the previous decomposition holds locally.

The previous result was proved in [13] and [24] for the Riemannian case, and then it was extended for metrics with signature in [65].

Definition 1.2.15 *A pseudo-Riemannian manifold (M, g) is called locally symmetric if $\nabla R = 0$, where R is the curvature of g.*

Although this is not the original definition of locally symmetric spaces, but rather the characterization achieved by E. Cartan, for the sake of simplicity it will be enough for the moment. We will study symmetric spaces and Cartan's Theorem in more detail in Section 2.1. We are now in position to enounce Berger's Theorem, the proof of which can be found with geometric arguments in [56].

Theorem 1.2.16 (Berger's Theorem) *Let (M, g) be a pseudo-Riemannian manifold of signature (r, s). If (M, g) is irreducible and non-locally symmetric, then the restricted holonomy group is one of the following:*

- $SO(r, s)$,
- $U(p, q)$, $r = 2p$, $s = 2q$,
- $SU(p, q)$, $r = 2p$, $s = 2q$,

- $Sp(p,q)$, $r=4p$, $s=4q$,
- $Sp(p,q)Sp(1)$, $r=4p$, $s=4q$,
- $SO(r,\mathbb{C})$, $r=s$,
- $Sp(p)SL(2,\mathbb{R})$, $r=s=2p$,
- $Sp(p,\mathbb{C})SL(2,\mathbb{C})$, $r=s=4p$,
- G_2, $r=0$, $s=7$,
- $G^*_{2(2)}$, $r=4$, $s=3$,
- $G_2^{\mathbb{C}}$, $r=s=7$,
- $Spin(7)$, $r=0$, $s=7$,
- $Spin(4,3)$, $r=s=4$,
- $Spin(7)^{\mathbb{C}}$, $r=s=8$.

The initial list of Berger [9, 10] was refined and completed by Bryant, Chi, Merkulov and Schwachhöfer (see [14, 21, 44]).

Note that in the Riemannian setting, the notions of irreducible manifolds and indecomposable manifolds coincide. Therefore, Theorems 1.2.14 and 1.2.16 provide a complete classification of non-locally symmetric Riemannian manifolds and their possible geometric structures. However, this is not the case when metrics with signature are considered. There is a gap between Theorem 1.2.16 (which deals with irreducible manifolds) and Theorem 1.2.14 (which refers to indecomposable manifolds). This problem can be solved by obtaining a classification of indecomposable representations of Lie algebras $\mathfrak{g}\subset\mathfrak{so}(r,s)$, but the difficulty of this problem is considerably higher than the case of irreducible representations. So far the solution is only known for Lorentzian manifolds and manifolds of index 2 (see [33]), making Holonomy Theory of pseudo-Riemannian manifolds still a field of intense research.

1.2.3 Geometric description of some G-structures

Theorem 1.2.16 suggests to study G-structures with G a group appearing in its list. In particular, in the present thesis we will be interested in the groups $U(p,q)$, $Sp(p,q)$ and $Sp(p,q)Sp(1)$, hence a more detailed description of these G-structures is needed. In this section we review some of their geometric features, with special interest when these G-structures are integrable. We will also study some G-structures whose structure group does not appear in Theorem 1.2.16, but which are still of great interest, namely para-Kähler, para-quaternion Kähler, Sasakian, and cosymplectic structures.

Pseudo-Kähler manifolds

For a detailed introduction to complex manifolds see [38, Ch. IX].

An *almost complex structure* on a manifold M of dimension $m=2n$ is a reduction of $\mathcal{L}(M)$ to structure group $GL(n,\mathbb{C})\subset GL(2n,\mathbb{R})$, seen as the stabilizer of the $(1,1)$-tensor on $\mathbb{R}^{2n}$

$$J_0=\begin{pmatrix}0 & -Id_n\\ Id_n & 0\end{pmatrix}.$$

Equivalently, an almost complex structure is a $(1,1)$-tensor field J on M satisfying $J^2=-Id$. This tensor field seen as a section of $\operatorname{End}(TM)$ provides an splitting of the complexified tangent bundle and the complexified cotangent bundle

$$T^cM=T^{1,0}\oplus T^{0,1},\qquad T^{*c}M=T^{*1,0}\oplus T^{*0,1},$$

corresponding to the eigenspaces of J with eigenvalues $\pm i$ respectively. The second splitting defines a bigraduation

$$\Omega^r(M,\mathbb{C}) = \bigoplus_{p+q=r} \Omega^{p,q}(M,\mathbb{C})$$

in the space of complex r-forms. A complex r-form belonging to $\Omega^{p,q}(M,\mathbb{C})$ is called of type (p,q). In the same way, a section of $T^{1,0}$ (resp. $T^{0,1}$) is called a complex vector field of type $(1,0)$ (resp. $(0,1)$).

The celebrated Theorem by Newlander and Nirenberg [48] asserts that the following statements are equivalent:

1. M is a complex manifold, that is, M admits an atlas $\{\mathcal{U}_\alpha\}$ of complex valued coordinates $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{C}^n$ with holomorphic transition functions.

2. M admits an integrable $GL(n,\mathbb{C})$-structure.

3. M admits a complex structure, that is, an almost complex structure J with vanishing Nijenhuis tensor field

$$N(X,Y) = J[JX,Y] + J[X,JY] + [X,Y] - [JX,JY].$$

For a complex manifold (M,J)

$$d(\Omega^{p,q}(M,\mathbb{C})) \subset \Omega^{p+1,q}(M,\mathbb{C}) \oplus \Omega^{p,q+1}(M,\mathbb{C}),$$

so that we have differential operators

$$\partial : \Omega^{p,q}(M,\mathbb{C}) \to \Omega^{p+1,q}(M,\mathbb{C}), \qquad \bar\partial : \Omega^{p,q}(M,\mathbb{C}) \to \Omega^{p,q+1}(M,\mathbb{C}),$$

defined by the corresponding projections. These satisfy $\partial^2 = 0$, $\bar\partial^2 = 0$, $\partial\circ\bar\partial + \bar\partial\circ\partial = 0$. A function $f : M \to \mathbb{C}$ is said *holomorphic* if $\bar\partial f = 0$. In the same way, a complex $(p,0)$-form ω is *holomorphic* if $\bar\partial\omega = 0$. A *holomorphic* vector field is a complex vector field Z of type $(1,0)$ such that $Z(f)$ is holomorphic whenever $f : M \to \mathbb{C}$ is holomorphic. An analogous definition can be made for anti-holomorphic functions, $(0,q)$-forms, and vector fields of type $(0,1)$. There is a Lie algebra isomorphism between the set of infinitesimal automorphisms of J (i.e. $\mathcal{L}_X J = 0$) and the set of holomorphic vector fields given by $X \mapsto \frac{1}{2}(X - iJX)$. Finally, a mapping $f : (M,J) \to (M',J')$ between complex manifolds is called *holomorphic* if $J' \circ f_* = f_* \circ J$.

A *pseudo-Hermitian metric* on (M,J) is a pseudo-Riemannian metric g such that

$$g(JX,JY) = g(X,Y),$$

or equivalently, a reduction to group structure $U(p,q)$, where the signature of g is $(2p,2q)$, $p+q = n$. In that case, (M,g,J) is called an almost pseudo-Hermitian manifold, and if J is complex then it is called a pseudo-Hermitian manifold.

Definition 1.2.17 *An almost pseudo-Hermitian manifold (M,g,J) is called pseudo-Kähler if one of the following equivalent conditions holds (see Proposition 1.2.12):*

(a) The $U(p,q)$-structure is integrable.

(b) $\nabla J = 0$, where ∇ is the Levi-Civita connection of g.

(c) The holonomy group of g is contained in $U(p,q)$.

A pseudo-Kähler manifold (M, g, J) is in particular a complex manifold with complex structure J. Moreover, it is a symplectic manifold with symplectic form $\omega(X, Y) = g(X, JY)$. This relation between symplectic, complex, and pseudo-Kähler manifolds can be read from

$$U(p, q) = O(2p, 2q) \cap GL(n, \mathbb{C}) = O(2p, 2q) \cap Sp(n, \mathbb{R}) = Sp(n, \mathbb{R}) \cap GL(n, \mathbb{C}).$$

Concerning the curvature of a pseudo-Kähler manifold, since the holonomy algebra of g is contained in $\mathfrak{u}(p, q)$, the curvature tensor field R and the Ricci tensor field Ric have the following symmetries:

$$\begin{aligned} &R_{JXY} + R_{XJY} = 0, \\ &R_{XY} JZ = J R_{XY} Z, \\ &Ric_{JXJY} = Ric_{XY}. \end{aligned}$$

A real form β is called of type $(1, 1)$ if $\beta(JX, JY) = \beta(X, Y)$. It is easy to see that in that case the complexification of β is a complex form of type $(1, 1)$. The map $b \mapsto \beta = b(\cdot, J\cdot)$ defines a linear isomorphism between the space of symmetric J-invariant bilinear forms (i.e., $b(JX, JY) = b(X, Y)$) and the space of real 2-forms of type $(1, 1)$, and the image of the Ricci tensor field under this isomorphism is called the *Ricci form* ρ of (M, g, J). It is a well-known result that the Ricci-form of a pseudo-Kähler manifold is closed.

Let $x \in M$, and let π be a non-degenerate 2-plane of T_xM. We say that π is complex if it is invariant by J. In that case we define $K_x(\pi) = R_{XJXXJX}$, where X is a unitary vector of π. The function K_x is called the *holomorphic sectional curvature* at x. If K_x is constant for every non-degenerate complex 2-plane of T_xM and for every $x \in M$ we say that (M, g) is of *constant holomorphic sectional curvature*. It is a well known result that in that case the curvature tensor field takes the form

$$\begin{aligned} R(X, Y, Z, W) = \frac{k}{4} \Big\{ &g(Y, Z) g(X, W) - g(X, Z) g(Y, W) - g(X, JZ) g(Y, JW) \\ &+ g(X, JW) g(Y, JZ) - 2 g(X, JY) g(Z, JW) \Big\}, \end{aligned}$$

where $k \in \mathbb{R}$ is the value of the holomorphic sectional curvature. It is straightforward to adapt the arguments from the well-known case of definite metrics to prove that two spaces of constant and equal holomorphic sectional curvature are locally holomorphically isometric (see for instance [6]). When (M, g, J) is in addition connected, simply-connected and complete, then it is called a *complex space form*. In that case (M, g, J) is flat or holomorphically isometric to the symmetric spaces $\mathbb{C}\mathrm{P}^n_p(k)$ if $k > 0$ or $\mathbb{C}\mathrm{H}^n_p(k)$ if $k < 0$, where

$$\mathbb{C}\mathrm{P}^n_p(k) = \frac{SU(n+1-p, p)}{S(U(n-p, p) \times U(1))}, \quad \mathbb{C}\mathrm{H}^n_p(k) = \frac{SU(n-p, p+1)}{S(U(n-p, p) \times U(1))}, \tag{1.1}$$

are endowed with a suitable metric such that its holomorphic sectional curvature is constant and equal to k. Spaces of constant holomorphic sectional curvature will be of great importance later.

Remark 1.2.18 *There is a diffeomorphism between $\mathbb{C}\mathrm{H}^n_p(k)$ and $\mathbb{C}\mathrm{P}^n_{n-p}(-k)$ (for $k < 0$) which is an isometry up to a change of sign. Therefore the cases $k > 0$ and $k < 0$ are equivalent for our purposes, and we can restrict ourselves to one of them.*

Para-Kähler manifolds

For a complete introduction on para-complex geometry see for instance [3] and [23].

Let $\widetilde{\mathbb{C}} = \mathbb{R} + e\mathbb{R}$ be the set of para-complex numbers, where e denotes the para-complex imaginary unit, i.e., $e^2 = 1$. An *almost para-complex structure* on a $2n$-dimensional manifold M is a reduction of $\mathcal{L}(M)$ to structure group $GL(n, \widetilde{\mathbb{C}})$ seen as the stabilizer of the $(1,1)$-tensor on $\mathbb{R}^{2n}$

$$J_0 = \begin{pmatrix} 0 & Id_n \\ Id_n & 0 \end{pmatrix}$$

inside $GL(2n, \mathbb{R})$. Equivalently, an almost para-complex structure is a $(1,1)$-tensor field J on M satisfying $J^2 = Id$, $J \neq Id$, and such that the eigenspaces of J_x, seen as an endomorphism of T_xM, corresponding to eigenvalues ± 1 have the same dimension for every $x \in M$. J provides an splitting of the para-complexified tangent bundle and the para-complexified cotangent bundle

$$T^cM = T^{1,0} \oplus T^{0,1}, \qquad T^{*c}M = T^{*1,0} \oplus T^{*0,1},$$

corresponding to the eigenspaces of J with eigenvalues $\pm e$ respectively. The second splitting defines a bigraduation

$$\Omega^r(M, \widetilde{\mathbb{C}}) = \bigoplus_{p+q=r} \Omega^{p,q}(M, \widetilde{\mathbb{C}})$$

in the space of para-complex r-forms. A para-complex r-form belonging to $\Omega^{p,q}(M, \widetilde{\mathbb{C}})$ is called of type (p,q). In the same way, a section of $T^{1,0}$ (resp. $T^{0,1}$) is called a para-complex vector field of type $(1,0)$ (resp. $(0,1)$).

An analogous result to Newlander-Nirenberg Theorem asserts that the following statements are equivalent:

1. (M, J) is para-complex, that is, there is an atlas $\{\mathcal{U}_\alpha\}$ of para-complex valued coordinates $\varphi_\alpha : \mathcal{U}_\alpha \to \widetilde{\mathbb{C}}^n$ with para-holomorphic transition functions.

2. The $GL(n, \widetilde{\mathbb{C}})$-structure is integrable.

3. The almost para-complex structure J is para-complex, i.e. it satisfies $N = 0$, where
$$N(X,Y) = J[JX,Y] + J[X,JY] - [X,Y] - [JX,JY].$$

It is worth recalling that one of the differences between complex and para-complex manifolds is that para-complex coordinates may not be real analytic. For a para-complex manifold (M, J)

$$d(\Omega^{p,q}(M, \widetilde{\mathbb{C}})) \subset \Omega^{p+1,q}(M, \widetilde{\mathbb{C}}) \oplus \Omega^{p,q+1}(M, \widetilde{\mathbb{C}}),$$

so that there are differential operators

$$\partial : \Omega^{p,q}(M, \widetilde{\mathbb{C}}) \to \Omega^{p+1,q}(M, \widetilde{\mathbb{C}}), \qquad \bar{\partial} : \Omega^{p,q}(M, \widetilde{\mathbb{C}}) \to \Omega^{p,q+1}(M, \widetilde{\mathbb{C}}),$$

defined by the corresponding projections, satisfying $\partial^2 = 0$, $\bar{\partial}^2 = 0$, $\partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0$. A function $f : M \to \widetilde{\mathbb{C}}$ is said *para-holomorphic* if $\bar{\partial} f = 0$. In the same way, a para-complex $(p,0)$-form ω is *holomorphic* if $\bar{\partial}\omega = 0$. A *para-holomorphic* vector field is a para-complex vector field Z of type $(1,0)$ such that $Z(f)$ is para-holomorphic whenever $f : M \to \widetilde{\mathbb{C}}$ is para-holomorphic. An analogous definition can be made for anti-para-holomorphic functions, vector fields of type $(0,1)$, and $(0,q)$-forms. There is a Lie algebra isomorphism between the set of infinitesimal automorphisms of J and the set of para-holomorphic vector fields given by $X \mapsto \frac{1}{2}(X + eJX)$. Finally, a map f between two para-complex manifolds (M, J) and (M', J') is called para-holomorphic if $f_* \circ J = J' \circ f_*$.

A *para-Hermitian metric* on (M, J) is a pseudo-Riemannian metric g such that

$$g(JX, JY) = -g(X, Y),$$

or equivalently, a reduction to structure group

$$GL(n, \mathbb{R}) = O(n, n) \cap GL(n, \widetilde{\mathbb{C}}) = \left\{ \begin{pmatrix} B & 0 \\ 0 & (B^{-1})^T \end{pmatrix} : B \in Gl(n, \mathbb{R}) \right\}.$$

In that case the signature of g is (n, n), and (M, g, J) is called an almost para-Hermitian manifold. If J is para-complex then (M, g, J) is called a para-Hermitian manifold.

Definition 1.2.19 *An almost para-Hermitian manifold (M, g, J) is called para-Kähler if one of the following equivalent conditions holds (see Proposition 1.2.12):*

(a) The $GL(n, \mathbb{R})$-structure is integrable.

(b) $\nabla J = 0$, where ∇ is the Levi-Civita connection of g.

(c) The holonomy group of g is contained in $GL(n, \mathbb{R})$.

A para-Kähler manifold (M, g, J) is in particular a para-complex manifold with para-complex structure J. Moreover, it is a symplectic manifold with symplectic form $\omega(X, Y) = g(X, JY)$. This relation between symplectic, para-complex, and para-Kähler manifolds can be read from

$$GL(n, \mathbb{R}) = O(n, n) \cap GL(n, \widetilde{\mathbb{C}}) = O(n, n) \cap Sp(n, \mathbb{R}) = Sp(n, \mathbb{R}) \cap GL(n, \widetilde{\mathbb{C}}).$$

Concerning the curvature of a para-Kähler manifold, since the holonomy algebra of g is contained in $\mathfrak{gl}(n, \mathbb{R})$, the curvature tensor field R and the Ricci tensor field Ric have the following symmetries:

$$\begin{aligned} R_{JXY} + R_{XJY} &= 0, \\ R_{XY} JZ &= J R_{XY} Z, \\ Ric_{JXJY} &= -Ric_{XY}. \end{aligned}$$

Real forms of type $(1, 1)$ are defined analogously to the complex case, and in the same way, the Ricci form $\rho(X, Y) = Ric_{XJY}$ is a closed form of type $(1, 1)$.

Let $x \in M$, and let π be a non-degenerate 2-plane of $T_x M$. We say that π is para-complex if it is invariant by J. In that case we define $K_x(\pi) = R_{XJXXJX}$, where X is a unitary vector of π. The function K_x is called the *para-holomorphic sectional curvature* at x. If K_x is constant for every non-degenerate para-complex 2-plane of $T_x M$ and for every $x \in M$, we say that (M, g) is of *constant para-holomorphic sectional curvature.* It is a well known result that in that case the curvature tensor field takes the form

$$\begin{aligned} R(X, Y, Z, W) = \frac{k}{4} \Big\{ & g(Y, Z) g(X, W) - g(X, Z) g(Y, W) + g(X, JZ) g(Y, JW) \\ & - g(X, JW) g(Y, JZ) + 2 g(X, JY) g(Z, JW) \Big\}, \end{aligned}$$

where $k \in \mathbb{R}$ is the value of the para-holomorphic sectional curvature. It is straightforward to adapt the arguments from the well-known case of definite metrics to prove that two spaces of constant and equal para-holomorphic sectional curvature are locally para-holomorphically isometric [28]. When (M, g, J) is in addition connected, simply-connected and complete, then it is called a *para-complex space form.* In that case (M, g, J) is flat or para-holomorphically isometric to the symmetric space

$$\widetilde{\mathbb{C}}\mathrm{P}^n(k) = \frac{SL(n+1, \mathbb{R})}{S(GL(n, \mathbb{R}) \times GL(1, \mathbb{R}))}, \tag{1.2}$$

endowed with a suitable metric such that its para-holomorphic sectional curvature is constant and equal to k. Spaces of constant para-holomorphic sectional curvature will be of great importance later.

Pseudo-quaternion Kähler manifolds

An *almost quaternionic structure* on a manifold M of dimension $m = 4n$ is a reduction of $\mathcal{L}(M)$ to structure group $GL(n,\mathbb{H})Sp(1)$, where $\mathbb{H}$ denotes the set of quaternions. The group $GL(n,\mathbb{H})Sp(1)$ can be seen as the stabilizer inside $GL(4n,\mathbb{R})$ of the three dimensional subspace of $End(\mathbb{R}^{4n})$ generated by

$$I_0 = \begin{pmatrix} 0 & -Id & 0 & 0 \\ Id & 0 & 0 & 0 \\ 0 & 0 & 0 & -Id \\ 0 & 0 & Id & 0 \end{pmatrix} \quad J_0 = \begin{pmatrix} 0 & 0 & -Id & 0 \\ 0 & 0 & 0 & Id \\ Id & 0 & 0 & 0 \\ 0 & -Id & 0 & 0 \end{pmatrix}$$

$$K_0 = \begin{pmatrix} 0 & 0 & 0 & -Id \\ 0 & 0 & -Id & 0 \\ 0 & Id & 0 & 0 \\ Id & 0 & 0 & 0 \end{pmatrix}.$$

Note that $\{I_0, J_0, K_0\}$ generates an algebra isomorphic to the imaginary quaternions. For this reason an almost quaternionic structure on M is equivalent to the existence of a 3-rank subbundle $Q \subset \mathrm{End}(M)$ such that there is a local basis $\{J_1, J_2, J_3\}$ satisfying

$$J_1^2 = J_2^2 = J_3^2 = -Id, \qquad J_1 J_2 = J_3.$$

Two local basis $\{J_1, J_2, J_3\}$ and $\{J_1', J_2', J_3'\}$ are related by $J_a' = \sum_{b=1}^{3} C_{ab} J_b$ for certain matrix $(C_{ab}) \in SO(3)$.

An *almost pseudo-quaternion Hermitian structure* on M is an almost quaternionic structure Q and a pseudo-Riemannian metric g of signature $(4p, 4q)$ such that

$$g(J_a X, Y) + g(X, J_a Y) = 0, \qquad a = 1, 2, 3,$$

that is, Q is a subbundle of $\mathfrak{so}(M)$. This is equivalent to a reduction to structure group $Sp(p,q)Sp(1)$. Let

$$\omega_a = g(\cdot, J_a \cdot), \qquad a = 1, 2, 3,$$

it is easily seen that the 4-form

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$$

is globally defined. This way, the group $Sp(p,q)Sp(1)$ can be seen as the stabilizer inside $SO(4p, 4q)$ of a 4-form on $\mathbb{R}^{4n}$ constructed from $\{I_0, J_0, K_0\}$ and the standard metric of signature $(4p, 4q)$ analogously to how Ω is constructed from J_1, J_2, J_3 and g.

Definition 1.2.20 *An almost pseudo-quaternion Hermitian manifold (M, g, Q) is called pseudo-quaternion Kähler if one of the following equivalent conditions holds (see Proposition 1.2.12):*

(a) The $Sp(p,q)Sp(1)$-structure is integrable.

(b) $\nabla\Omega = 0$, where ∇ is the Levi-Civita connection of g.

(c) The holonomy group of g is contained in $Sp(p,q)Sp(1)$.

(d) For every local basis $\{J_1, J_2, J_3\}$ of Q

$$\nabla J_a = \sum_{b=1}^{3} c_{ab} J_b, \qquad a = 1, 2, 3,$$

with (c_{ab}) a matrix of 1-forms in $\mathfrak{so}(3)$.

The fact that (d) is equivalent to (a), (b) and (c) is actually a result by Ishihara [35, 36]. However, since so far the present author has not found a complete proof of this result, we exhibit a simple proof here.

Proof of Ishihara's Theorem. Suppose that (M, g, Q) has holonomy contained in $Sp(p,q)Sp(1)$. Let $\mathcal{P}$ be the holonomy bundle, which is a reduction of $\mathcal{O}(M)$ to structure group $Sp(p,q)Sp(1)$. The group $Sp(1)$ can be seen as the group $S^3 \subset \mathbb{H}$ of quaternions of norm 1 (that is, $q\bar{q} = 1$), so that it acts on the imaginary quaternions $\mathrm{Im}(\mathbb{H})$ as

$$\begin{array}{rccc} L_q : & \mathrm{Im}(\mathbb{H}) & \to & \mathrm{Im}(\mathbb{H}) \\ & z & \mapsto & qz\overline{q}. \end{array}$$

Under the identification $\mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$, L_q is an element of $SO(3)$ determining a $1:2$ covering $Sp(1) \to SO(3)$. We consider the associated bundle

$$Q = \mathcal{P} \times_{Sp(p,q)Sp(1)} \mathbb{R}^3,$$

where $Sp(p,q)Sp(1)$ acts on $\mathbb{R}^3$ through the action of $Sp(1)$ described above. This vector bundle is seen as a three rank subbundle of $\mathrm{End}(M)$ by

$$\begin{array}{ccc} P \times_{Sp(n)Sp(1)} \mathbb{R}^3 & \hookrightarrow & P \times_{Sp(n)Sp(1)} \mathrm{End}(\mathbb{R}^{4n}) \\ [v, \xi] & \mapsto & [v, \xi_1 I_0 + \xi_2 J_0 + \xi_3 K_0]. \end{array}$$

It is easy to see that a local section σ of $\mathcal{P}$ determines a local basis $J_1 = [\sigma, e_1]$, $J_2 = [\sigma, e_2]$, $J_3 = [\sigma, e_3]$ of Q which satisfies

$$J_1^2 = J_2^2 = J_3^2 = -Id, \qquad J_1 J_2 = J_3,$$

that is, Q is an almost quaternionic structure. Let τ be a curve in M and $S_0 = [u_0, \xi] \in Q$. The horizontal lift of τ to S_0 in $\mathrm{End}(M)$ is just the curve $[u_t, \xi]$, where u_t is the horizontal lift of τ to u_0. By definition u_t is contained in $\mathcal{P}$, so that $[u_t, \xi]$ is contained in Q. This means that Q is invariant by parallel transport. Therefore, let τ_0^t denote the parallel transport along τ from $t = 0$ to t, there is a matrix $(C_{ab}(t)) \in SO(3)$ such that

$$\tau_0^t(J_a) = \sum_{b=1}^{3} C_{ab}(t) J_b, \qquad a = 1, 2, 3.$$

differentiating at $t = 0$ we find that

$$\nabla_X J_a = \sum_{b=1}^{3} c_{ab} J_b, \qquad a = 1, 2, 3,$$

where $X = \dot{\tau}(t = 0)$, and $c_{ab} = \frac{d}{dt}\big|_{t=0} C_{ab}(t) \in \mathfrak{so}(3)$. Conversely, suppose that

$$\nabla_X J_a = \sum_{b=1}^{3} c_{ab} J_b, \qquad a = 1, 2, 3.$$

Since $\nabla g = 0$ we have that

$$\nabla_X \omega_a = \sum_{b=1}^{3} c_{ab} \omega_b, \qquad a = 1, 2, 3,$$

and thus a straightforward computation shows that $\nabla \Omega = 0$. ■

Regarding the curvature, we will say that an algebraic curvature tensor R is of type $\mathfrak{sp}(p,q)$ if it sits in $\odot^2\mathfrak{sp}(p,q)$ when seen as an element of $\odot^2\mathfrak{so}(4p,4q)$. This kind of tensors satisfies the following symmetries:

$$\begin{aligned} &R_{J_aXY} + R_{XJ_aY} = 0, && a = 1,2,3, \\ &R_{XY}J_aZ = J_aR_{XY}Z, && a = 1,2,3, \\ &Ric = 0. \end{aligned}$$

The proof of the following proposition can be found in [2].

Proposition 1.2.21 *The curvature of a pseudo-quaternion Kähler manifold* (M,g,Q) *decomposes as*

$$R = \nu_q R^0 + R^{\mathfrak{sp}(p,q)},$$

where $\nu_q = \frac{\mathrm{s}}{16n(n+2)}$, R^0 *is four times the curvature of the pseudo-quaternionic hyperbolic space (of the corresponding signature)*

$$\begin{aligned} R^0_{XYZW} \quad = \quad & g(X,Z)g(Y,W) - g(Y,Z)g(X,W) + \sum_a \{g(J_aX,Z)g(J_aY,W) \\ & -g(J_aY,Z)g(J_aX,W) + 2g(X,J_aY)g(Z,J_aW)\}, \end{aligned} \tag{1.3}$$

and $R^{\mathfrak{sp}(p,q)}$ *is of type* $\mathfrak{sp}(p,q)$. *In particular* (M,g,Q) *is Einstein.*

Let $x \in M$, and let $Z \in T_xM$ with $g(Z,Z) \neq 0$. We consider the 4-dimensional subspace $V(Z) = \mathrm{Span}\{Z, J_1Z, J_2Z, J_3Z\} \subset T_xM$. Let $\pi \subset V(Z)$ be a non-degenerate 2-plane and $\{X,Y\}$ an orthonormal basis of π, if $K_x(Z)(\pi) = R_{XYXY}$ is constant for every π, we call $K_x(Z)$ the quaternionic sectional curvature with respect to Z at x. We say that (M,g,Q) has *constant quaternionic sectional curvature* if $K_x(Z)$ is constant for every $Z \in T_xM$ and every $x \in M$. It is a well known result that in that case the curvature tensor field takes the form

$$R = \frac{k}{4}R^0,$$

where R^0 is given by (1.3) and k is the value of the quaternionic sectional curvature. It is straightforward to adapt the arguments from the well-known case of definite metrics to prove that two spaces of constant and equal quaternionic sectional curvature are locally isometric preserving their pseudo-quaternion Kähler structures (see for instance [52]). When (M,g,Q) is in addition connected, simply-connected and complete, then it is called a *quaternion space form*. In that case (M,g,Q) is flat or isometric (preserving the pseudo-quaternion Kähler structures) to the symmetric spaces $\mathbb{H}\mathrm{P}^n_p(k)$ if $k > 0$ or $\mathbb{H}\mathrm{H}^n_p(k)$ if $k < 0$, where

$$\mathbb{H}\mathrm{P}^n_p(k) = \frac{Sp(p,n+1-p)}{Sp(p,n-p)Sp(1)}, \quad \mathbb{H}\mathrm{H}^n_p(k) = \frac{Sp(p+1,n-p)}{Sp(p,n-p)Sp(1)}, \tag{1.4}$$

endowed with a suitable metric such that its quaternionic sectional curvature is constant and equal to k. Spaces of constant quaternionic sectional curvature will be of great importance later.

Remark 1.2.22 *There is a diffeomorphism between* $\mathbb{H}\mathrm{H}^p_s(k)$ *and* $\mathbb{H}\mathrm{P}^n_{n-p}(-k)$ *(for* $k < 0$*) which is an isometry up to a change of sign. Therefore the cases* $k > 0$ *and* $k < 0$ *are equivalent for our purposes, and we can restrict ourselves to one of them.*

Although pseudo-hyper-Kähler manifolds will not give many interesting results in this thesis, this kind of manifolds are intimate related to pseudo-quaternion Kähler manifolds and will be briefly treated later. For this reason we recall its definition. An *almost pseudo-hyper-complex structure* on M, $\dim M = 4n$, is a three rank subbundle of $\mathrm{End}(M)$ which admits a global basis $\{J_1, J_2, J_3\}$ satisfying

$$J_1^2 = J_2^2 = J_3^2 = -Id, \qquad J_1 J_2 = J_3,$$

or equivalently a reduction of $\mathcal{L}(M)$ to group $GL(n, \mathbb{H})$, which is seen as the common stabilizer of I_0, J_0, K_0 inside $GL(4n, \mathbb{R})$. An *almost pseudo-hyper-Hermitian structure* on M is an almost pseudo-hyper-complex structure J_1,J_2, J_3 and a pseudo-Riemannian metric of signature $(4p, 4q)$ such that

$$g(J_a X, Y) + g(X, J_a Y) = 0, \qquad a = 1, 2, 3,$$

or equivalently a reduction to structure group $Sp(p, q)$. We thus have

Definition 1.2.23 *An almost pseudo-hyper-Hermitian manifold is called pseudo-hyper-Kähler if one of the following equivalent conditions holds (see Proposition 1.2.12):*

(a) The $Sp(p, q)$-structure is integrable.

(b) $\nabla J_a = 0$ for $a = 1, 2, 3$, where ∇ is the Levi-Civita connection of g.

(c) The holonomy group of g is contained in $Sp(p, q)$.

It is evident that the curvature tensor field of a pseudo-hyper-Kähler manifold is of type $\mathfrak{sp}(p, q)$.

Para-quaternion Kähler manifolds

Let $\widetilde{\mathbb{H}}$ denote the set of para-quaternions (also known as split-quaternions). An *almost para-quaternionic structure* on a manifold M of dimension $m = 4n$ is a reduction of $\mathcal{L}(M)$ to structure group $GL(n, \widetilde{\mathbb{H}})Sp(1, \mathbb{R})$. The group $GL(n, \widetilde{\mathbb{H}})Sp(1, \mathbb{R})$ can be seen as the stabilizer inside $GL(4n, \mathbb{R})$ of the three dimensional subspace of $End(\mathbb{R}^{4n})$ generated by

$$I_0 = \begin{pmatrix} 0 & -Id & 0 & 0 \\ Id & 0 & 0 & 0 \\ 0 & 0 & 0 & -Id \\ 0 & 0 & Id & 0 \end{pmatrix} \quad J_0 = \begin{pmatrix} 0 & Id & 0 & 0 \\ Id & 0 & 0 & 0 \\ 0 & 0 & 0 & Id \\ 0 & 0 & Id & 0 \end{pmatrix}$$

$$K_0 = \begin{pmatrix} -Id & 0 & 0 & 0 \\ 0 & Id & 0 & 0 \\ 0 & 0 & -Id & 0 \\ 0 & 0 & 0 & Id \end{pmatrix}.$$

Note that $\{I_0, J_0, K_0\}$ generates an algebra isomorphic to the set of imaginary para-quaternions. For this reason an almost para-quaternionic structure on M is equivalent to the existence of a three rank subbundle $Q \subset \mathrm{End}(M)$ such that there is a local basis $\{J_1, J_2, J_3\}$ satisfying

$$J_1^2 = -Id, \qquad J_2^2 = J_3^2 = Id, \qquad J_1 J_2 = J_3.$$

Two local basis $\{J_1, J_2, J_3\}$ and $\{J_1', J_2', J_3'\}$ are related by $J_a' = \sum_{b=1}^{3} C_{ab} J_b$ for certain matrix $(C_{ab}) \in SO(1, 2)$.

A pseudo-Riemannian manifold of signature (r, s) is said *strongly oriented* if the bundle of orthonormal frames can be reduced to the connected component $SO_0(r, s)$ (since $SO(r, s)/SO_0(r, s)$ is discrete there always exists a strongly oriented cover of M).

An *almost para-quaternion Hermitian structure* on a strongly oriented pseudo-Riemannian manifold (M, g) of signature $(2n, 2n)$ is an almost para-quaternionic structure Q such that

$$g(J_a X, Y) + g(X, J_a Y) = 0, \qquad a = 1, 2, 3,$$

that is, Q is a subbundle of $\mathfrak{so}(M)$. This is equivalent to a reduction to structure group $Sp(n,\mathbb{R})Sp(1,\mathbb{R}) \subset SO_0(2n,2n)$. Let

$$\omega_a = g(\cdot, J_a\cdot), \qquad a = 1,2,3,$$

it is easily seen that the 4-form

$$\Omega = \omega_1 \wedge \omega_1 - \omega_2 \wedge \omega_2 - \omega_3 \wedge \omega_3$$

is globally defined. This way, the group $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$ can be seen as the stabilizer inside $SO_0(2n,2n)$ of a 4-form on $\mathbb{R}^{4n}$ constructed from $\{I_0, J_0, K_0\}$ and the standard metric of signature $(2n,2n)$ analogously to how Ω is constructed from J_1, J_2, J_3 and g.

Definition 1.2.24 *An almost para-quaternion Hermitian manifold (M,g,Q) is called para-quaternion Kähler if one of the following equivalent conditions holds (see Proposition 1.2.12):*

(a) The $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$-structure is integrable.

(b) $\nabla\Omega = 0$, where ∇ is the Levi-Civita connection of g.

(c) The holonomy of g is contained in $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$.

(d) For every local basis $\{J_1, J_2, J_3\}$ of Q

$$\nabla J_a = \sum_{b=1}^{3} c_{ab} J_b, \qquad a = 1,2,3,$$

with (c_{ab}) a matrix of 1-forms in $\mathfrak{so}(1,2)$.

The fact that (d) is equivalent to (a), (b) and (c) can be proved analogously to the pseudo-quaternion Kähler case.

Regarding the curvature, we will say that an algebraic curvature tensor R is of type $\mathfrak{sp}(n,\mathbb{R})$ if it sits in $\odot^2\mathfrak{sp}(n,\mathbb{R})$ when seen as an element of $\odot^2\mathfrak{so}(2n,2n)$. This kind of tensors satisfy the following symmetries:

$$\begin{aligned} &R_{J_aXY} + R_{XJ_aY} = 0, \qquad && a = 1,2,3,\\ &R_{XY}J_aZ = J_aR_{XY}Z, \qquad && a = 1,2,3,\\ &Ric = 0. \end{aligned}$$

The proof of the following proposition can be found in [2].

Proposition 1.2.25 *A para-quaternion Kähler manifold (M,g,Q) has curvature tensor field*

$$R = \nu_q R^0 + R^{\mathfrak{sp}(n,\mathbb{R})},$$

where $\nu_q = \frac{s}{16n(n+2)}$, R^0 is four times the curvature of the para-quaternionic hyperbolic space (of the corresponding signature)

$$\begin{aligned} R^0_{XYZW} \;=\; & g(X,Z)g(Y,W) - g(Y,Z)g(X,W) - \sum_a \epsilon_a \{g(J_aX,Z)g(J_aY,W) \\ & -g(J_aY,Z)g(J_aX,W) + 2g(X,J_aY)g(Z,J_aW)\}, \end{aligned} \tag{1.5}$$

with $(\epsilon_1,\epsilon_2,\epsilon_3) = (-1,1,1)$ and $R^{\mathfrak{sp}(n,\mathbb{R})}$ is of type $\mathfrak{sp}(n,\mathbb{R})$. In particular (M,g,Q) is Einstein.

Let $x \in M$, and let $Z \in T_xM$ with $g(Z,Z) \neq 0$. We consider the 4-dimensional subspace $V(Z) = \mathrm{Span}\{Z, J_1Z, J_2Z, J_3Z\} \subset T_xM$. Let $\pi \subset V(Z)$ be a non-degenerate 2-plane and $\{X, Y\}$ an orthonormal basis of π, if $K_x(Z)(\pi) = R_{XYXY}$ is constant for every π, we call $K_x(Z)$ the para-quaternionic sectional curvature with respect to Z at x. We say that (M, g, Q) has *constant para-quaternionic sectional curvature* if $K_x(Z)$ is constant for every $Z \in T_xM$ and every $x \in M$. It is a well known result that in that case the curvature tensor field takes the form

$$R = \frac{k}{4} R^0,$$

where R^0 is given by (1.5) and k is the value of the para-quaternionic sectional curvature. It is straightforward to adapt the arguments from the well-known case of definite metrics to prove that two spaces of constant and equal para-quaternionic sectional curvature are locally isometric preserving their para-quaternion Kähler structures (see for instance [62]). When (M, g, Q) is in addition connected, simply-connected and complete, then it is called a *para-quaternion space form*. In that case (M, g, Q) is flat or isometric (preserving the para-quaternion Kähler structures) to the symmetric space

$$\widetilde{\mathbb{H}}\mathrm{P}^n(k) = \frac{Sp(n+1,\mathbb{R})}{Sp(n,\mathbb{R})Sp(1,\mathbb{R})} \tag{1.6}$$

endowed with a suitable metric such that its para-quaternionic sectional curvature is constant and equal to k. Spaces of constant para-quaternionic sectional curvature will be of great importance later.

Although again para-hyper-Kähler manifolds will not give many interesting results in this thesis, this kind of manifolds are intimate related to para-quaternion Kähler manifolds and will be briefly treated later. For this reason we recall its definition. An *almost para-hyper-complex structure* on M, $\mathrm{dim}M = 4n$, is a three rank subbundle of $\mathrm{End}(M)$ which admits a global basis $\{J_1, J_2, J_3\}$ satisfying

$$J_1^2 = -Id, \qquad J_2^2 = J_3^2 = Id, \qquad J_1J_2 = J_3,$$

or equivalently a reduction of $\mathcal{L}(M)$ to group $GL(n, \widetilde{\mathbb{H}})$, which is seen as the common stabilizer inside $GL(4n, \mathbb{R})$ of I_0, J_0, K_0. An *almost para-hyper-Hermitian structure* on M is an almost para-hyper-complex structure J_1, J_2, J_3 and a pseudo-Riemannian metric of signature $(2n, 2n)$ such that

$$g(J_aX, Y) + g(X, J_aY) = 0, \qquad a = 1, 2, 3,$$

or equivalently a reduction to structure group $Sp(n, \mathbb{R})$. We thus have

Definition 1.2.26 *An almost para-hyper-Hermitian manifold (M, g, J_1, J_2, J_3) is called para-hyper-Kähler if one of the following equivalent conditions holds (see Proposition 1.2.12):*

(a) The $Sp(n, \mathbb{R})$-structure is integrable.

(b) $\nabla J_a = 0$ for $a = 1, 2, 3$, where ∇ is the Levi-Civita connection of g.

(c) The holonomy group of g is contained in $Sp(n, \mathbb{R})$.

It is evident that the curvature tensor field of a para-hyper-Kähler manifold is of type $\mathfrak{sp}(n, \mathbb{R})$.

Sasakian and cosymplectic manifolds

For a complete introduction to Sasakian and cosymplectic structures, and for detailed proofs see [12, 58].

Definition 1.2.27 *1. An almost contact structure on a $2n+1$-dimensional manifold M is a triple (ϕ, ξ, η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field and η is a 1-form, such that*

$$\eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi^2 = -\mathrm{id} + \eta \otimes \xi.$$

2. Let g be a pseudo-Riemannian metric on M, (ϕ, ξ, η, g) is called an almost contact metric structure if (ϕ, ξ, η, g) is an almost contact structure and

$$g(\xi, \xi) = \varepsilon \in \{\pm 1\}, \quad \eta = \varepsilon \xi^\flat, \quad g(\phi X, \phi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y),$$

for any vector fields X, Y.

An almost contact metric structure on M is equivalent to a reduction to structure group $U(p,q) \times \{1\}$. Here, $U(p,q) \times \{1\}$ is the subgroup of $O(2p+1, 2q)$ or $O(2p, 2q+1)$ (depending on the value of ε) stabilizing $\xi_0 = e_{2n+1}$ and

$$\phi_0 = \begin{pmatrix} J_0 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\{e_1, \ldots, e_{2n+1}\}$ is the canonical basis of $\mathbb{R}^{2n+1} = \mathbb{R}^{2n} \oplus \mathbb{R}$, and J_0 is the standard complex structure on $\mathbb{R}^{2n}$. Here $\mathbb{R}^{2n+1}$ is assumed to be endowed with the scalar product $\langle\, , \rangle = \langle\, , \rangle_{\mathbb{R}^{2n}} + \varepsilon e^{2n+1} \otimes e^{2n+1}$, where $\langle\, , \rangle_{\mathbb{R}^{2n}}$ is the standard scalar product of signature (p,q) on $\mathbb{R}^{2n}$. We define *the fundamental* 2*-form* associated to the almost contact metric structure (ϕ, ξ, η, g) as $\Phi = g(\cdot, \phi\cdot)$. In addition, in analogy with the Nijenhuis tensor field for complex manifolds we define (see [12])

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

Definition 1.2.28 *An almost contact metric structure (ϕ, ξ, η, g) is called cosymplectic if one of the following equivalent conditions hold:*

(a) The $U(p,q) \times \{1\}$-structure is integrable.

(b) (ϕ, ξ, η, g) satisfies $[\phi, \phi] = 0$, $d\eta = 0$, and $d\Phi = 0$.

(c) $\nabla\phi = 0$ (which implies $\nabla\eta = 0$ and $\nabla\xi = 0$).

(d) The holonomy group of g is contained in $U(p,q) \times \{1\}$.

Definition 1.2.29 *An almost contact metric structure (ϕ, ξ, η, g) is said to be Sasakian if one of the following equivalent conditions hold:*

1. (ϕ, ξ, η, g) satisfies $[\phi, \phi] + 2\eta \otimes \xi = 0$, and $d\eta = \Phi$.

2. $(\nabla_X \phi)Y = g(X, Y)\xi - \varepsilon \eta(Y) X$.

Note that Sasakian geometry is not an integrable geometry, in the sense that the holonomy group is not contained in $U(p,q) \times \{1\}$, that is, the $U(p,q) \times \{1\}$-structured determined by the almost contact metric structure is not integrable. This fact will require a slight modification when classifying homogeneous Sasakian structures in Chapter 4. In order to do that a few words about the intrinsic torsion of an almost contact metric structure must be said (for an excellent introduction to the intrinsic torsion of a G-structure see [53, 26]). As we have seen above, the $U(p,q) \times \{1\}$-structure determined by the almost contact metric structure (ϕ, ξ, η, g) is integrable if and only if $\nabla\phi = 0$ (or equivalently if and only if $\nabla\Phi = 0$). We can thus see the tensor field $\nabla\phi$ (or $\nabla\Phi$) as the obstruction for (ϕ, ξ, η, g) to be cosymplectic. Therefore, one can study the possible non-integrable geometries an almost contact metric manifold can present by studying the possible tensor fields $\nabla\phi$ (or $\nabla\Phi$) arising. This can be done considering the vector space $V = \mathbb{R}^{2n+1}$ endowed with the standard almost contact metric structure described above. We then take the space $\mathcal{C}(V)$ of tensors of type $(0,3)$ with the same symmetries as $\nabla\Phi$, that is,

$$T_{XYZ} = -T_{XYZ} = -T_{X\phi Y\phi Z} + \eta(Y)T_{X\xi Z} + \eta(Z)T_{XY\xi}.$$

$\mathcal{C}(V)$ is an $U(p,q) \times \{1\}$-module in a natural way, so that one can decompose it into irreducible V-submodules. This was achieve in [22] for the Riemannian case obtaining twelve irreducible submodules, and the case of metrics with signature is obtained by a straightforward adaptation. Each of this submodules determines a class of geometric structures. The class corresponding to the so called α-Sasakian structures is given by the submodule

$$\mathcal{C}_6(V) = \{T \in \mathcal{C}(V) / T_{XYZ} = \alpha\varepsilon\left(\langle X, Y\rangle\eta(Z) - \langle X, Z\rangle\eta(Y)\right), \alpha \in \mathbb{R}\},$$

from which Sasakian structures corresponds to $\alpha = 1$. This will be used in section 4.2.6.

We finally recall the notion of ϕ-sectional curvature. Let

$$D_x = \{X \in T_xM,\ \eta(X) = 0\}.$$

If $X \in T_xM$ is a unitary vector, then X and ϕX span a non-degenerate plane π, and hence we can consider the sectional curvature $K_x(\pi) = R_{X\phi XX\phi X}$ of that plane. If K_x is constant for all unitary vectors $X \in D_x$ and every $x \in M$, then we say that M is of *constant ϕ-sectional curvature*. In that case the curvature tensor field takes the form

$$\begin{aligned} 4R(X,Y)Z &= (k + 3\varepsilon)\{g(Y,Z)X - g(X,Z)Y\} \\ &+ (\varepsilon k - 1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \\ &+ (k - \varepsilon)\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+ g(\phi Y, Z)\phi X + g(\phi Z, X)\phi Y - 2g(\phi X, Y)\phi Z\}, \end{aligned}$$

where k is the constant value of the ϕ-sectional curvature.

1.3 Homogeneous spaces and the canonical connection

Let G be a Lie group and H a subgroup. It is a classic problem to try to endow the quotient G/H with a "good" differentiable structure, that is a differentiable structure such that $\pi : G \to G/H$ is a submersion. This is not always possible, and the main problem is to ensure that the quotient topology is Hausdorff. It is easy to see that the quotient G/H is Hausdorff if and only if H is a closed subgroup of G. In fact we have the following well known result (see [38, Ch. I]).

Theorem 1.3.1 *Let G be a Lie group and H a closed subgroup of G. Then there is a unique differentiable structure on G/H such that the action of G on G/H is $\mathcal{C}^\infty$, that is, the mapping $G \times G/H \to G/H$, $(a, bH) \mapsto abH$ is $\mathcal{C}^\infty$. In particular $\pi : G \to G/H$ is a submersion. In addition,* $\mathrm{dim}G/H = \mathrm{dim}G - \mathrm{dim}H$.

Note that for every two points $aH, bH \in G/H$ the differentiable mapping induced by the left translation by ab^{-1} takes the point bH to aH, implying that the orbit of a point aH by the action of G is the whole G/H. This fact suggests the following equivalent definitions.

Definition 1.3.2 (Definition of homogeneous space 1) *A homogeneous space is a quotient G/H of a Lie group G by a closed subgroup H, endowed with the unique differentiable structure making $\pi : G \to G/H$ a submersion.*

Definition 1.3.3 (Definition of homogeneous space 2) *A manifold M is homogeneous if there is a Lie group G acting on the left on M, such that the action is $\mathcal{C}^\infty$ and transitive.*

We shall denote by $L_a : M \to M$ the action of an element $a \in G$. We shall also often denote by $a \cdot p$ the action of $a \in G$ on a point p. The equivalence between the previous two definitions is given by the following construction. Let G/H be the quotient of a Lie group G by a closed subgroup H, then G acts on G/H by left translations, and as we have seen this action is transitive. Conversely, let M be a manifold on which G acts transitively. We define the *isotropy group* at a point $p \in M$ as the subgroup

$$H = \{a \in G / a \cdot p = p\}.$$

H is a closed subgroup of G and it is a straightforward computation to see that the map

$$\begin{array}{rcl} G/H & \to & M \\ aH & \mapsto & a \cdot p \end{array}$$

defines a diffeomorphism between G/H and M. Note that in both cases $G \to G/H$ is a principal bundle with structure group H. A group G is said to act *effectively* on M if the subgroup

$$N = \{a \in G / \, L_a = Id_M\}$$

only contains the neutral element of G. Since N is a normal subgroup of G, we can always assume that G acts effectively on M replacing G by G/N.

Definition 1.3.4 *A pseudo-Riemannian manifold (M, g) is called homogeneous if there is a Lie group G of isometries acting transitively on the left on M.*

If a connected pseudo-Riemannian manifold is homogeneous then the isometry group $Isom(M)$ and $Isom_0(M, g)$ acts transitively on M, where $Isom_0(M, g)$ is the connected component of $Isom(M, g)$ containing the identity. Moreover, G can be identify with a Lie subgroup of $Isom(M, g)$.

Remark 1.3.5 *It can be proved (see [38, Ch. I]) that a homogeneous manifold always admits a real analytic structure, such that the action $G \times G/H \to G/H$ and the projection $G \to G/H$ are real analytic maps. Although most of the time we will only be concerned about $\mathcal{C}^\infty$ structures and maps, we will make use of this fact when necessary.*

As we have seen, homogeneous spaces enjoy a large group of internal symmetries. For that reason they constitute a distinguished class of spaces on which the study of pseudo-Riemannian geometry is especially rich and varied. However, this privileged position is often paid with their rigidity. Weakening Definition 1.3.3 we can obtain a larger and less rigid class of spaces which still share most of the desirable properties of homogeneous spaces.

Definition 1.3.6 *A pseudo-Riemannian manifold (M, g) is called locally homogeneous if the pseudo-group of local isometries acts transitively on (M, g), that is, if for every two points $p, q \in M$ there are neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of p and q respectively, and an isometry $f : \mathcal{U} \to \mathcal{V}$ taking p to q.*

The following definition will be central for the rest of this dissertation.

Definition 1.3.7 *A homogeneous space G/H is called reductive if the Lie algebra $\mathfrak{g}$ of G can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace, that is, $\mathrm{Ad}_h(\mathfrak{m}) \subset \mathfrak{m}$ for every $h \in H$.*

The condition $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$ implies $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and the converse holds if H is connected. Note that we can identify the subspace $\mathfrak{m}$ with T_oM, where $o = H \in G/H$ is the origin of G/H, as follows: let $X \in \mathfrak{m}$, we consider the one parameter group $\exp(tX)$, and identify X with X^*_o, where X^* is the *fundamental vector field* associated to X defined by

$$X^*_p = \left.\frac{d}{dt}\right|_{t=0} L_{\exp(tX)}(p), \qquad p \in G/H.$$

By definition it is easy to see that

$$[X^*, Y^*] = -[X, Y]^*, \qquad (\mathrm{Ad}_g(X))^*_{L_g(p)} = (L_g)_{*p}(X^*_p).$$

Proposition 1.3.8 *Every homogeneous Riemannian space (M, g) is reductive.*

Proof. We fix the origin $o \in M$. Let $M = G/H$, and let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H respectively. We can suppose that G acts effectively. For every $X \in \mathfrak{g}$ we consider the associated vector field X^*. It is clear that $\mathfrak{h}$ consists of those $X \in \mathfrak{g}$ such that $X^*_o = 0$. Let ∇ be the Levi-Civita connection of g. Since X^* is a Killing vector field, the operator $A_{X*} = \mathcal{L}_{X^*} - \nabla_X = -\nabla X$ is skew-symmetric, so that $A_{X^*|o} \in \mathfrak{so}(T_oM)$. Let B denote the Killing form of $\mathfrak{so}(T_oM)$, we consider the symmetric bilinear form ϕ on $\mathfrak{g}$ defined as

$$\phi(X, Y) = -B(A_{X^*|o}, A_{Y^*|o}).$$

Since g is positive definite, B is negative definite so that $\phi(X, X) = 0$ implies that $A_{X^*|o} = 0$. Therefore, if $X \in \mathfrak{h}$ and $\phi(X, X) = 0$ we have $A_{X^*|o}$ and $X^*_o = 0$, whence $X^* = 0$. Since G acts effectively we obtain $X = 0$. This proves that ϕ is definite on $\mathfrak{h}$. In addition, let $h \in H$, for every $X, Y \in \mathfrak{g}$

$$\begin{aligned}
\phi(\mathrm{Ad}_h(X), \mathrm{Ad}_h(Y)) &= -B\left(A_{\mathrm{Ad}_h(X)^*|o}, A_{\mathrm{Ad}_h(Y)^*|o}\right) \\
&= -B\left(A_{L_{h*}(X^*_o)}, A_{L_{h*}(Y^*_o)}\right) \\
&= -\sum_k g\left(\nabla_{e_k} L_{h*}(X^*_o), \nabla_{e_k} L_{h*}(Y^*_o)\right) \\
&= -\sum_k g\left(\nabla_{e_k} X^*_o, \nabla_{e_k} Y^*_o\right) \\
&= \phi(X, Y),
\end{aligned}$$

since L_{h*} is an isometry, where $\{e_k\}$ is any orthonormal basis of T_oM. This shows that ϕ is $\mathrm{Ad}(H)$-invariant. Finally, we take the orthogonal complement $\mathfrak{m} = \mathfrak{h}^\perp$ of $\mathfrak{h}$ with respect to ϕ. By construction $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{m}$ is $\mathrm{Ad}(H)$-invariant. ■

We now turn to define a very special invariant connection: the so called *canonical connection.* But before we need some technical preliminaries.

Lemma 1.3.9 *For every $X \in \mathfrak{X}(M)$ there is a unique vector field $\widetilde{X}$ on $\mathcal{L}(M)$ such that*

1. *$\widetilde{X}$ is invariant by the right action of $GL(m, \mathbb{R})$.*
2. *$\mathcal{L}_{\widetilde{X}} \theta = 0$.*
3. *$\pi_*(\widetilde{X}_u) = X_{\pi(u)}$ for every $u \in \mathcal{L}(M)$.*

Moreover, for every $\widetilde{X}$ on $\mathcal{L}(M)$ satisfying (a) and (b), there is a unique vector field X on M satisfying (c). The vector field $\widetilde{X}$ is called the natural lift of X.

Proof. Let f_t be the one parameter group of local transformations of X, we consider the set of induced local maps $\widetilde{f}_t$ on $\mathcal{L}(M)$. The vector field $\widetilde{X}$ is defined as the vector field generating $\widetilde{f}_t$. It is straightforward to prove uniqueness and properties (a), (b) and (c). ∎

Let $M = K/H$ be a homogeneous space. We denote by $o \in M$ the coset H, which will be called the origin of M. We consider the map

$$\begin{array}{ccc} H & \to & Aut(T_oM) \\ h & \mapsto & (L_h)_{*,o}. \end{array}$$

It is immediate to see that this map is a group homomorphism, which will be called the *linear isotropy representation.* By definition, the isotropy representation is faithful if and only if the action of K is effective, or equivalently if the induced action of K on $\mathcal{L}(M)$ is free. Hereafter we suppose that K is connected and the isotropy representation is faithful. We will also assume that there is a G-structure $P(M,G)$ which is invariant by the induced action of K on $\mathcal{L}(M)$, that is, for every $a \in K$ we have an induced map $\widetilde{L_a} : P \to P$. Note that all the subsequent results can be applied to pseudo-Riemannian homogeneous spaces by taking the bundle of orthonormal references $\mathcal{O}(M)$ as $P(M,G)$. Let $u_0 \in P$ such that $\pi(u_0) = o$. We say that a linear connection on $P(M,G)$ is *invariant* by the action of K if it is invariant by $\widetilde{L_a}$ for every $a \in K$.

Identifying T_oM with $\mathbb{R}^m$ through the isomorphism $u_0 : \mathbb{R}^m \to T_oM$, we can see the linear isotropy representation as the homomorphism

$$\begin{array}{cccl} \lambda: & H & \to & G \\ & h & \mapsto & \lambda(h) = u_0^{-1} \circ (L_h)_{*,o} \circ u_0. \end{array} \tag{1.7}$$

We shall also denote by λ the corresponding homomorphism of Lie algebras $\lambda : \mathfrak{h} \to \mathfrak{g}$. Let $X \in \mathfrak{k}$, consider the one parameter subgroup $\exp(tX)$ of K, which determines a one parameter group of transformations $f_t = L_{\exp(tX)}$ of M. We will also denote by X the vector field associated to f_t, that is $X_p = \frac{d}{dt}\big|_{t=0} f_t(p)$, $p \in K/H$.

Theorem 1.3.10 (Wang) *Let $P(M,G)$ be an invariant G-structure on a reductive homogeneous space K/H with reductive decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$. Then, there is a one to one correspondence between invariant connections on P and linear maps $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{g}$ such that*

$$\Lambda_{\mathfrak{m}}(\mathrm{Ad}_h(X)) = \mathrm{Ad}_{\lambda(h)}(\Lambda_{\mathfrak{m}}(X)), \quad X \in \mathfrak{m}, h \in H.$$

The correspondence can be read from

$$\omega_{u_0}(\widetilde{X}) = \left\{ \begin{array}{ll} \Lambda_{\mathfrak{m}}(X) & X \in \mathfrak{m} \\ \lambda(X) & X \in \mathfrak{h} \end{array} \right. ,$$

where ω is the 1-form of the invariant connection.

Proof. The correspondence (see [38]) is given by

$$\Lambda(X) = \left\{ \begin{array}{lll} \lambda(X) & \text{if} & X \in \mathfrak{h} \\ \Lambda_{\mathfrak{m}}(X) & \text{if} & X \in \mathfrak{m} \end{array} \right.$$

∎

It is obvious that the linear map $\Lambda_{\mathfrak{m}} = 0$ satisfies the condition in Theorem 1.3.10. The corresponding invariant connection thus enjoys a distinguished position among invariant connections and will be of great importance in this thesis.

Definition 1.3.11 *The invariant connection corresponding to the linear map* $\Lambda_{\mathfrak{m}} = 0$ *is called the canonical connection associated to the reductive decomposition* $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$.

The canonical connection associated a certain reductive decomposition has a geometric characterization which is often very useful (see [38, Ch. X, Corollary 2.5]).

Proposition 1.3.12 *Let* $P(M, G)$ *be a* K*-invariant* G*-structure on a homogeneous space* K/H*. The canonical connection associated to a reductive decomposition* $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ *is the unique* K*-invariant connection on* K *with the following property: let* $f_t = \exp(tX)$ *be the one parameter subgroup of* K *generated by* $X \in \mathfrak{m}$*, and let* $\widetilde{f}_t$ *be the induced transformations of* P*, then the orbit* $\widetilde{f}_t(u_0)$ *is horizontal, where* $u_0 \in \pi^{-1}(o)$.

Corollary 1.3.13 *Let* $X \in \mathfrak{m}$*, consider the curve* $\gamma_t = L_{\exp(tX)}(o)$ *in* K/H*. The parallel transport along* γ_t *from* o *to* γ_s *coincides with the differential of* $f_s = L_{\exp(sX)}$.

Proposition 1.3.14 *[38] The torsion and the curvature of the canonical connection associated to a reductive decomposition* $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ *are given by:*

$$\widetilde{T}(X, Y)_o = [X, Y]_{\mathfrak{m}}, \qquad X, Y \in \mathfrak{m},$$
$$(\widetilde{R}(X, Y)Z)_o = [[X, Y]_{\mathfrak{h}}, Z], \qquad X, Y, Z \in \mathfrak{m},$$

where the subindex $\mathfrak{h}$ *and* $\mathfrak{m}$ *indicates projection on the corresponding subspace with respect to the reductive decomposition. In addition, let* $\widetilde{\nabla}$ *denote the covariant derivative with respect to the canonical connection, then*

$$\widetilde{\nabla}\widetilde{R} = 0, \qquad \widetilde{\nabla}\widetilde{T} = 0.$$

Proposition 1.3.15 *If a tensor field on* M *is invariant by the action of* K*, then it is parallel with respect to the canonical connection.*

Proof. Let J be a K-invariant tensor field of type (k, l) on M. Let $X \in T_oM$, under the identification of T_oM and $\mathfrak{m}$ we consider $f_t = L_{\exp(tX)}$ and the curve $\gamma_t = f_t(o)$. By Proposition 1.3.12 the horizontal lift of γ_t to $u_0 \in \pi^{-1}(o)$ is $\bar{\gamma}_t = \widetilde{f}_t(u_0)$. Let

$$E = P \times_G \left(\otimes^k (\mathbb{R}^m)^*\right) \otimes \left(\otimes^l \mathbb{R}^m\right)$$

be the associated bundle of tensor fields of type (k, l), and $J_o = [u_0, J_0]$. The horizontal lift of γ_t to $[u_0, J_0]$ in E is given by $[\widetilde{f}_t(u_0), J_0]$. But since J is K-invariant one has $[\widetilde{f}_t(u_0), J_0] = J_{\gamma_t}$. We obtain by the definition of the covariant derivative on E that $(\widetilde{\nabla} J)_o = 0$. Since the action of K is transitive and $\widetilde{\nabla}$ is invariant we deduce that $\widetilde{\nabla} J = 0$. ■

The converse of the previous Proposition also holds and is part of Kiričenko's Theorem, which will be treated in Chapter 2.

Chapter 2

Ambrose-Singer connections and homogeneous spaces

2.1 Symmetric spaces and Cartan's Theorem

In this section we recall the definition and some properties of symmetric and locally symmetric spaces. We also present Cartan's Theorem which characterizes these spaces in terms of the covariant derivative of the curvature. This is the starting point and the motivation for Ambrose-Singer's Theorem, which generalizes Cartan's characterization to homogeneous spaces. For an extensive study of symmetric and locally symmetric spaces see [34].

Let (M, g) be a connected pseudo-Riemannian space, and ∇ its Levi-Civita connection. For every $x \in M$ we consider the symmetry s_x which inverts geodesics of ∇ through x. s_x is a diffeomorphism from a neighborhood U of x onto itself such that $s_x \circ s_x$ is the identity transformation, that is s_x is involutive, and has x as an isolated fixed point.

Definition 2.1.1 *(M, g) is called locally symmetric if s_x is a local isometry for every $x \in M$. If moreover s_x extends to a global isometry of M then (M, g) is called (globally) symmetric.*

It is a well-known result (see [38, Ch. XI]) that, by composing symmetries s_x along broken geodesics, the isometry group of a symmetric space (M, g) acts transitively on M, so that a symmetric space is a homogeneous space. More precisely, let G denote the identity component of the group of isometries, then G acts transitively on M, and we can write $M = G/H$, where H is the isotropy group at a point $o \in M$. Let $s_o : M \to M$ be the symmetry at $o \in M$, we consider the map $\sigma : G \to G$, $g \mapsto s_o \circ g \circ s_o$, which is obviously an involutive automorphism of G. Let G^σ denote the fixed point set of σ and G_0^σ its identity component, then $G_0^\sigma \subset H \subset G^\sigma$. The differential $\sigma_* : \mathfrak{g} \to \mathfrak{g}$ is also an involutive automorphism of Lie algebras, so that it has eigenvalues ± 1. A Lie algebra with such an involutive automorphism is called a *symmetric Lie algebra.* It is not hard to see that $\mathfrak{h}$ is the eigenspace associated to $+1$. Denoting by $\mathfrak{m}$ the eigenspace associated to -1, we have that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

In particular G/H is reductive. Identifying $T_oM \simeq \mathfrak{m}$, the pseudo-Riemannian metric g induces a non-degenerate symmetric bilinear form B on $\mathfrak{m}$, which is necessarily $\mathrm{Ad}(H)$-invariant. Conversely, let G be a Lie group with an involutive automorphism σ, and let H be a closed subgroup sitting between G_0^σ and G^σ. Let B be an $\mathrm{Ad}(H)$-invariant non-degenerate symmetric bilinear form on $\mathfrak{m}$ (which is the -1-eigenspace of σ_*), then G/H can be endowed with a G-invariant pseudo-Riemannian metric g making $(G/H, g)$ a symmetric space. In fact, under the identification $\mathfrak{m} \simeq T_o(G/H)$, g is the G-invariant metric on G/H with $g_o = B$. In addition, let $\pi : G \to G/H$ denote the canonical projection, the map $s_o \circ \pi = \pi \circ \sigma$ determines an isometric involution with fixed point o, which can be transported by the action of G obtaining isometric involutions at every point of G/H.

Theorem 2.1.2 (E. Cartan) *A pseudo-Riemannian manifold (M, g) is locally symmetric if and only if $\nabla R = 0$, where ∇ is the Levi-Civita connection of g and R its curvature tensor field. If M is simply-connected and complete then (M, g) is symmetric if and only if $\nabla R = 0$.*

Although we don't include the proof of this theorem (see for instance [38, Ch. XI]), it is very interesting to point out some of the arguments used, specially when comparing with the arguments used in the proof of Ambrose-Singer's Theorem presented in the next section. Starting from a symmetric space $(G/H, \sigma, g)$, we consider the canonical connection associated to the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ and $\mathfrak{m}$ are the eigenspaces of σ corresponding to ± 1 as before. One proves that this connection has vanishing torsion, and since g is G-invariant it is also metric, so that it coincides with the Levi-Civita connection of g. This implies that $\nabla R = 0$ by the properties of the canonical connection. On the other hand, let (M, g) be such that $\nabla R = 0$. Fixing a point $x \in M$ one considers

$$\mathfrak{g} = \mathfrak{hol} \oplus T_x M,$$

where $\mathfrak{hol}$ is the Lie algebra of the holonomy group *Hol* of g. Note that since $\nabla R = 0$, R is invariant by the action of *Hol*, and so is by the action of $\mathfrak{hol}$. Therefore, the brackets

$$\begin{aligned} [A, B] &= AB - BA, \qquad A, B \in \mathfrak{hol}, \\ [A, X] &= A \cdot X, \qquad A \in \mathfrak{hol}, X \in T_x M, \\ [X, Y] &= R_{XY}, \qquad X, Y \in T_x M, \end{aligned}$$

endow $\mathfrak{g}$ with a Lie algebra structure such that $\sigma_* : \mathfrak{g} \to \mathfrak{g}$ with $\sigma_{|T_x M} = -id$ and $\sigma_{|\mathfrak{hol}} = id$ is an involutive automorphism, that is $(\mathfrak{g}, \sigma_*)$ is a symmetric Lie algebra. Consider the simply-connected Lie group G with Lie algebra $\mathfrak{g}$, and its connected Lie subgroup H with Lie group $\mathfrak{hol}$. σ_* induces an involutive automorphism $\sigma : G \to G$ such that H is the connected component of G^σ, hence H is closed in G and we can take the homogeneous space G/H. Finally, the G-invariant pseudo-Riemannian metric $\bar{g}$ inherited from g at $T_x M$ makes $(G/H, \sigma, \bar{g})$ a symmetric space, which is moreover locally isometric to (M, g).

2.2 Ambrose-Singer and Kiričenko's Theorems

As we have seen, Cartan's Theorem characterizes (locally) symmetric spaces as pseudo-Riemannian manifolds whose curvature tensor is covariantly constant. Moreover, when the manifold is globally symmetric, this approach allows us to recover a coset representation of the manifold. In this section we present Ambrose-Singer's Theorem, which generalizes Cartan's Theorem to the more general framework of homogeneous spaces. Under suitable topological conditions, this result characterizes homogeneous spaces by the existence of a metric connection $\widetilde{\nabla}$ with respect to which the curvature tensor field of the metric and the torsion of $\widetilde{\nabla}$ have vanishing covariant derivative. Furthermore, it provides a method to recover a coset representation of the manifold, which, at least at a Lie algebra level, is achieved in terms of an elementary construction. This topic will be studied in the next section. Ambrose-Singer's Theorem is completed by Kiričenko's Theorem, which extends the theory to manifolds with a geometric structure given by a set of tensor fields.

Theorem 2.2.1 (Ambrose-Singer) *Let (M, g) be a connected, simply-connected and complete pseudo-Riemannian manifold. The following are equivalent:*

(a) (M,g) *is a reductive homogeneous pseudo-Riemannian manifold.*

(b) (M,g) *admits a linear connection* $\widetilde{\nabla}$ *satisfying*

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} S = 0, \tag{2.1}$$

where $S = \nabla - \widetilde{\nabla}$, ∇ *is the Levi-Civita connection of* g, *and* R *its curvature tensor field.*

This theorem was proved by Ambrose and Singer in [4] for Riemannian metrics, and later in [31] it was generalized to metrics of arbitrary signature.

Proposition 2.2.2 *Equations* (2.1) *are equivalent to*

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} \widetilde{R} = 0, \qquad \widetilde{\nabla} \widetilde{T} = 0,$$

where $\widetilde{R}$ *and* $\widetilde{T}$ *are the curvature and torsion tensor fields of* $\widetilde{\nabla}$ *respectively.*

Proof. The equivalence follows by direct calculation from the relations

$$\widetilde{T}_X Y = S_Y X - S_X Y,$$

$$\widetilde{R}_{XY} = R_{XY} + [S_X, S_Y] + S_{\widetilde{T}_X Y}.$$

■

Before enouncing and proving Kiričenko's Theorem we will need the following technical Lemma (see [60] and the proof therein).

Lemma 2.2.3 *Let* M *be a connected and simply-connected manifold of dimension* m. *Let* $X_1, \ldots, X_m$ *be vector fields such that*

(a) $X_1, \ldots, X_m$ *are complete.*

(b) $X_1, \ldots, X_m$ *are linearly independent at every point* $p \in M$.

(c) $[X_i, X_j] = \sum_{k=1}^{m} c_{ij}^k X_k$, *with* c_{ij}^k *constant.*

Then, fixing a point $p \in M$, M *has a unique structure of Lie group such that* p *is the neutral element and* $\{X_1, \ldots, X_m\}$ *is a basis of left-invariant vector fields.*

Theorem 2.2.4 (Kiričenko) *Let* (M,g) *be a connected, simply-connected and complete pseudo-Riemannian manifold with a geometric structure defined by a set of tensor fields* $P_1, \ldots, P_n$. *The following are equivalent:*

(a) (M,g) *is reductive homogeneous such that* $P_1, \ldots, P_n$ *are invariant.*

(b) (M,g) *admits a linear connection* $\widetilde{\nabla}$ *satisfying*

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} S = 0, \qquad \widetilde{\nabla} P_i = 0, \tag{2.2}$$

for $i = 1, \ldots, n$, *where* $S = \nabla - \widetilde{\nabla}$, ∇ *is the Levi-Civita connection of* g, *and* R *its curvature tensor field.*

Kiričenko's Theorem appears for the first time in [37], although Sekigawa in [54] had obtained the same result for almost Hermitian manifolds. Nevertheless, the proof appearing in the original paper by Kiričenko is incomplete, and despite the fact that this result has been extensively used since then, there is not a complete proof of it in the literature as far as the author knows. For this reason we present here a detailed proof of this theorem. Many of the arguments used therein are inherited from those used in the proof of Ambrose-Singer's theorem appearing in [60].

Proof of Kiričenko's Theorem. Part I. Let (M, g) be a reductive homogeneous pseudo-Riemannian manifold with invariant tensor fields $P_1, \ldots, P_n$. Let G be a Lie group acting transitively by isometries on (M, g) and preserving $P_1, \ldots, P_n$, and let H be the isotropy group at some point $p \in M$. We take $\widetilde{\nabla}$ the canonical connection with respect to a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. By Proposition 1.3.14 we have that $\widetilde{\nabla}\widetilde{R} = 0$ and $\widetilde{\nabla}\widetilde{T} = 0$, where $\widetilde{R}$ and $\widetilde{T}$ are the curvature and torsion tensor fields of $\widetilde{\nabla}$. As seen in Proposition 2.2.2 those equations imply that $\widetilde{\nabla}R = 0$ and $\widetilde{\nabla}S = 0$. Finally, making use of Proposition 1.3.15, since g and $P_1, \ldots, P_n$ are invariant by the action of G we obtain $\widetilde{\nabla}g = 0$ and $\widetilde{\nabla}P_i = 0$, $i = 1, \ldots, n$. ∎

Remark 2.2.5 *Since S in invariant by the action of a Lie group G acting transitively on (M, g), it is completely determined by its value at the origin $o \in M$. In fact, by the properties of the canonical connection*

$$(\widetilde{\nabla}_{\alpha^*}\beta^*)_o = -[\alpha, \beta]^*_o, \qquad \alpha, \beta \in \mathfrak{g},$$

whence

$$(S_{\alpha^*}\beta^*)_o = (\nabla_{\beta^*}\alpha^*)_o = -(A_{\alpha^*}\beta^*)_o.$$

This means that

$$(S_o)_X Y = -(A_{\alpha^*})_o Y, \qquad X, Y \in T_oM,$$

where α is the unique element in $\mathfrak{m}$ such that $\alpha^*_o = X$

Proof of Kiričenko's Theorem II.

Suppose that (M, g) admits a linear connection $\widetilde{\nabla}$ satisfying (2.2). Let $\mathcal{O}(M)$ be the bundle of orthonormal frames of (M, g), we fix a point $x \in M$ and reference $u_0 \in \mathcal{O}(M)$ with $\pi(u_0) = x$. Let H_i be the stabilizer of $u_0^*(P_i)_x$ inside $O(p, q)$, $i = 1, \ldots, n$, and $H = \cap_i H_i$. The tensor fields P_i determine a reduction Q of $\mathcal{O}(M)$ to group H such that $u_0 \in Q$. We take the holonomy bundle $\widetilde{\mathcal{P}}(u_0)$ of $\widetilde{\nabla}$ at u_0, which is a reduction of Q to group $Hol^{\widetilde{\nabla}} \subset H$, since $\widetilde{\nabla}P_i = 0$, $i = 1, \ldots, n$. We denote by $\mathfrak{h}$ and $\mathfrak{hol}^{\widetilde{\nabla}}$ the Lie algebras of H and $Hol^{\widetilde{\nabla}}$ respectively.

Let $\{A_1, \ldots, A_r\}$ be a basis of $\mathfrak{hol}^{\widetilde{\nabla}}$, we consider the associated fundamental vector fields $A_1^*, \ldots, A_r^*$, which are complete in Q. Let $\{e_1, \ldots, e_m\}$ be the canonical basis of $\mathbb{R}^m$, we take the associated standard vector fields $B_1 = B(e_1), \ldots, B_m = B(e_m)$. These vector fields are also complete in Q since $\widetilde{\nabla}$ is complete (see [38, Vol. I, p. 140]). Moreover, it is easy to see that the vector fields $A_1, \ldots, A_r$, $B_1, \ldots, B_m$ restricted to $\widetilde{\mathcal{P}}(u_0)$ are also complete and determine an absolute parallelism on $\widetilde{\mathcal{P}}(u_0)$.

Let ω be the connection 1-form of $\widetilde{\nabla}$ on the principal bundle $Q \to M$, and let θ be the contact form. We denote by Ω and Θ the curvature and torsion forms of $\widetilde{\nabla}$. Then for every $i, j = 1, \ldots, m$

$$\begin{aligned} \Theta(B_i, B_j) &= d\theta(B_i, B_j) \\ &= B_i(\theta(B_j)) - B_j(\theta(B_i)) - \theta([B_i, B_j]) \\ &= -\theta([B_i, B_j]), \end{aligned}$$

and

$$\begin{aligned} \Omega(B_i, B_j) &= d\omega(B_i, B_j) \\ &= B_i(\omega(B_j)) - B_j(\omega(B_i)) - \omega([B_i, B_j]) \\ &= -\omega([B_i, B_j]), \end{aligned}$$

whence the vertical and horizontal parts of $[B_i, B_j]$ are $-\Omega(B_i, B_j)^*$ and $-B\left(\Theta(B_i, B_j)\right)$. We can thus write

$$[B_i, B_j] = -B\left(\Theta(B_i, B_j)\right) - \Omega(B_i, B_j)^*.$$

In addition, by the properties of the fundamental vector fields (recall that the action of H on Q is on the right) it is evident that

$$\begin{aligned}[A_k^*, A_l^*] &= [A_k, A_l]^*,\\ [A_k, B_i] &= B(A_k(e_i)).\end{aligned}$$

for $k, l = 1, \ldots, r$ and $i = 1, \ldots, m$.

On the other hand, let $\widetilde{R}$ and $\widetilde{T}$ the curvature and torsion tensor fields of $\widetilde{\nabla}$, for every $u \in Q$ and every horizontal vector $\bar{X}$ in T_uQ we have

$$\begin{aligned}\bar{X}\left(\Theta(B_i, B_j)\right) &= u^{-1}\left(\left(\widetilde{\nabla}_X \widetilde{T}\right)(X_i, X_j)\right) = 0,\\ \bar{X}\left(\Omega(B_i, B_j)\right) &= u^{-1}\left(\left(\widetilde{\nabla}_X \widetilde{R}\right)(X_i, X_j)\right) = 0,\end{aligned}$$

where $X, X_i, X_j \in T_{\pi(u)}M$ are the projections of X, B_i, B_j respectively. We deduce that for every $i, j = 1, \ldots, m$, the functions $\Theta(B_i, B_j)$ and $\Omega(B_i, B_j)$ with values in $\mathbb{R}^m$ and $\mathfrak{h}$ respectively are constant on $\widetilde{\mathcal{P}}(u_0)$. Therefore, the brackets of the vector fields $A_k^*, B_i,\ i = 1, \ldots, m,\ k = 1, \ldots, r$, have constant coefficients with respect to the basis $\{A_k^*, B_i,\ i = 1, \ldots, m,\ k = 1, \ldots, r\}$, hence they span a finite dimensional subalgebra of $\mathfrak{X}(Q)$.

Let $\bar{G}$ be the universal cover of $\widetilde{\mathcal{P}}(u_0)$, we consider the vector fields $\bar{A}_k^*, \bar{B}_i,\ i = 1, \ldots, m,\ k = 1, \ldots, r$, on $\bar{G}$ defined by

$$\rho_*(\bar{A}_k^*) = A_k^*, \qquad \rho_*(\bar{B}_i) = B_i,$$

where $\rho : \bar{G} \to \widetilde{\mathcal{P}}(u_0)$ is the covering map. These vector fields are complete and determined an absolute parallelism. In addition their brackets have constant coefficients. Hence, making use of Lemma 2.2.3, $\bar{G}$ can be endowed with a Lie group structure with $\bar{e} \in \rho^{-1}(u_0)$ as the neutral element, and with $\{\bar{A}_k^*, \bar{B}_i,\ i = 1, \ldots, m,\ k = 1, \ldots, r\}$ spanning its Lie algebra $\bar{\mathfrak{g}}$. Note that $\{\bar{A}_k^*,\ k = 1, \ldots, r\}$ itself spans a subalgebra $\bar{\mathfrak{g}}_0 \subset \bar{\mathfrak{g}}$, whose corresponding connected Lie subgroup of $\bar{G}$ is denoted by $\bar{G}_0$.

Lemma 2.2.6 *M is diffeomorphic to $\bar{G}/\bar{G}_0$*

Proof. Let $\pi : \widetilde{\mathcal{P}}(u_0)$ be the projection of the holonomy bundle of $\widetilde{\nabla}$, the map $\pi_1 = \pi \circ \rho : \bar{G} \to M$ determines a fibration on M. Taking its exact homotopy sequence

$$\begin{array}{ccccccccc} \ldots & \stackrel{\pi_{1*}}{\to} & \Pi_1(M, y) & \stackrel{\partial_*}{\to} & \Pi_0(\pi_1^{-1}(y), \bar{b}) & \stackrel{i_*}{\to} & \Pi_0(\bar{G}, \bar{b}) & \stackrel{\pi_{1*}}{\to} & \Pi_0(M, g) \\ & & \| & & & & \| & & \| \\ & & 0 & & & & 0 & & 0 \end{array}$$

we deduce that $\Pi_0(\pi_1^{-1}(y), \bar{b}) = 0$, that is, the fibers of π_1 are connected. Since π_1 is continuous, its fibers are also closed. In addition, $\pi_{1*}(\bar{A}_k^*) = 0$ for $k = 1, \ldots, r$, hence the fibers are tangent to $\bar{\mathfrak{g}}_0$. We conclude that the fibers are the integral submanifolds of the involutive distribution $\bar{\mathfrak{g}}_0$, and can be thus represented as classes $\bar{a}\bar{G}_0$, where $\bar{a} \in \bar{G}$. Therefore, we have a $\mathcal{C}^\infty$ map $\pi_2 : \bar{G}/\bar{G}_0 \to M$, $\bar{a}\bar{G}_0 \mapsto \pi_1(\bar{a})$. π_2 is obviously bijective, and its differential is an isomorphism at every point, hence it is a diffeomorphism. ■

Let now $y \in M$ and $v = (y; v_1, \ldots, m) \in \widetilde{\mathcal{P}}(u_0)$, we can write

$$v = \left(\bar{b}\bar{G}_0; (\pi_{1*})_{\bar{b}}(\bar{B}_{1\bar{b}}), \ldots, (\pi_{1*})_{\bar{b}}(\bar{B}_{m\bar{b}})\right),$$

where $\rho(\bar{b}) = v$. We thus have

Lemma 2.2.7 *The maps*

$$L_{\bar a}: \begin{array}{ccc} M & \to & M \\ \bar b\bar G_0 & \mapsto & \bar a\bar b\bar G_0 \end{array}$$

induce transformations $\widetilde{L}_{\bar a} : \widetilde{\mathcal{P}}(u_0) \to \widetilde{\mathcal{P}}(u_0)$.

Proof. Let $\mathcal{L}_{\bar a}$ denote the left multiplication by $\bar a$ in $\bar G$, then $L_{\bar a} \circ \pi_1 = \pi_1 \circ \mathcal{L}_{\bar a}$. Let $y = \bar b\bar G_0$ and $v = \rho(\bar b) \in \widetilde{\mathcal{P}}(u_0)$. For all $i = 1, \ldots, m$ we have

$$\begin{aligned}(L_{\bar a *})_y \circ (\pi_1 *)_{\bar b}(\bar B_{i\bar b}) &= (\pi_{1*})_{\bar a\bar b} \circ (\mathcal{L}_{\bar a} *)_y(\bar B_{i\bar b}) \\ &= (\pi_{1*})_{\bar a\bar b}(\bar B_{i\bar a\bar b}),\end{aligned}$$

since $\bar B_i$ is left invariant. Taking into account that

$$\{\bar b\bar G_0; (\pi_{1*})_{\bar b}(\bar B_{1\bar b}), \ldots, (\pi_{1*})_{\bar b}(\bar B_{m\bar b})\} = \{y; v_1, \ldots, v_m\}$$

is a reference in $\widetilde{\mathcal{P}}(u_0)$ for every point of M, we conclude that $\widetilde{L}_{\bar a}$ is a transformation of $\widetilde{\mathcal{P}}(u_0)$ for every $\bar a \in \bar G$. ∎

We have thus proved that $\bar G$ acts on $\widetilde{\mathcal{P}}(u_0)$. Moreover, this action is transitive since

$$\begin{aligned}\widetilde{L}_{\bar b_1\bar b_2^{-1}}(\rho(\bar b_2)) &= \widetilde{L}_{\bar b_1\bar b_2^{-1}}\left(\bar b_2\bar G_0; (\pi_{1*})_{\bar b_2}(\bar B_{1\bar b_2}), \ldots, (\pi_{1*})_{\bar b_2}(\bar B_{m\bar b_2})\right) \\ &= \left\{\bar b_1\bar G_0; (\pi_{1*})_{\bar b_1}(\bar B_{m\bar b_1}), \ldots, (\pi_{1*})_{\bar b_1}(\bar B_{m\bar b_1})\right\} \\ &= \rho(\bar b_1).\end{aligned}$$

Note that $\widetilde{\mathcal{P}}(u_0) \subset Q$, so that the maps $L_{\bar a}$ act as isometries of (M, g) preserving the tensor fields P_i, $i = 1, \ldots, n$. Let now $\bar a$ be an element in the isotropy group $\bar K$ of the reference $u_0 \in \widetilde{\mathcal{P}}(u_0)$, that is, $\widetilde{L}_{\bar a}(u_0) = u_0$, then

$$L_{\bar a}(\pi(u_0)) = \pi(u_0),$$

and

$$L_{\bar a *} \circ (\pi_{1*})_{\bar e}(\bar B_{i\bar e}) = \bar B_{i\bar e}, \qquad i = 1, \ldots, m.$$

This means that $L_{\bar a}$ is an isometry of M fixing the point $x = \pi(u_0)$ and such that its differential at x is the identity, whence $L_{\bar a}$ is the identity transformation of M. Therefore, $\bar K$ is the kernel of the group homomorphism $\bar a \mapsto L_{\bar a}$, so that it is a normal subgroup of $\bar G$. We thus obtain that $\widetilde{\mathcal{P}}(u_0) = \bar G/\bar K$ is a Lie group acting transitively on M by isometries preserving $P_1, \ldots, P_n$.

Finally, the Lie algebra $\mathfrak{g} = T_{\bar e}\widetilde{\mathcal{P}}(u_0)$ of $G = \widetilde{\mathcal{P}}(u_0)$ can be decomposed as $\mathfrak{g} = \mathfrak{hol}^{\widetilde{\nabla}} \oplus \mathfrak{m}$, where $\mathfrak{hol}^{\widetilde{\nabla}}$ is spanned by $\{(A_k^*)_{u_0}, k = 1, \ldots, r\}$ and $\mathfrak{m}$ is spanned by $\{(B_i)_{u_0}, i = 1, \ldots, m\}$. We show that this decomposition is reductive. Consider the following maps

$$\rho : \bar G \to G,$$

$$p : \bar G \to \bar G/\bar K,\ \bar a \mapsto \bar a\bar K,$$

$$I : \bar G/\bar K \to G,\ \bar a\bar K \mapsto \widetilde{L}_{\bar a}(u_0).$$

It is easy to see that I is a diffeomorphism. Let K be the isotropy group of x with respect to the action of G. We have

$$L_{\bar a}(x) = L_{\bar a}(\bar e\bar G_0) = \bar a\bar G_0,$$

so that $\bar a \in \bar G_0$ if and only if $L_{\bar a} \in K$. We thus have that the diffeomorphism $I \circ p$ identifies $\bar G_0$ with K. In particular K is connected. On the other hand, the following diagram is commutative

$$\begin{array}{l} \overline{G} \xrightarrow{p} \overline{G}/\overline{K} \\ \rho \downarrow \swarrow I \\ \widetilde{P}\,(u) \end{array}$$

so that $\rho_* = I_* \circ p_*$. This means that $\mathfrak{hol}^{\widetilde{\nabla}} = \rho_*(\bar{\mathfrak{g}}_0) = (I \circ p)_*(\bar{\mathfrak{g}}_0)$ is contained in the Lie algebra of K, and counting dimensions we see that $\mathfrak{hol}^{\widetilde{\nabla}}$ equals the Lie algebra of K. It is obvious that $[\mathfrak{hol}^{\widetilde{\nabla}}, \mathfrak{m}] \subset \mathfrak{m}$, and since K is connected we have that $\mathfrak{m}$ is $\mathrm{Ad}(K)$-invariant. ■

Definition 2.2.8 *Let (M, g) be a pseudo-Riemannian manifold.*

(i) A linear connection $\widetilde{\nabla}$ on (M, g) satisfying (2.1) *will be called an Ambrose-Singer connection, or AS-connection for short.*

(ii) If (M, g) is moreover endowed with a geometric structure defined by tensor fields $P_1, \ldots, P_n$, a linear connection $\widetilde{\nabla}$ satisfying (2.2) *will be called an Ambrose-Singer-Kiričenko-connection or ASK-connection for short.*

2.3 Homogeneous structures

Definition 2.3.1 *Let (M, g) be a pseudo-Riemannian manifold with Levi-Civita connection ∇. Let $\widetilde{\nabla}$ be an AS-connection on (M, g). The tensor field $S = \nabla - \widetilde{\nabla}$ is called a homogeneous pseudo-Riemannian structure, or a homogeneous structure for short.*

In the previous section we have seen that a homogeneous pseudo-Riemannian manifold admits an AS-connection, and thus a homogeneous structure, whenever it is reductive. Conversely, a pseudo-Riemannian manifold admitting a homogeneous structure is a reductive homogeneous pseudo-Riemannian space under suitable topological conditions. Dropping those topological assumptions one only obtains that (M, g) is locally homogeneous. An analogous situation holds when a geometric structure is present. This result and its converse will be treated in Chapter 3. It is worth noting that, under the mentioned topological conditions, the proof of Ambrose-Singer's Theorem (or Kiričenko's Theorem) provides a method to construct a homogeneous pseudo-Riemannian manifold represented as a coset G/H starting from an AS-connection (or from a homogeneous structure S). We now study this construction in detail.

Let V be a vector space and let

$$K : V \wedge V \to \mathrm{End}(V),$$

$$T : V \to \mathrm{End}(V),$$

be morphisms.

Definition 2.3.2 *The pair (K, T) is called an infinitesimal model if the following properties are satisfied:*

$$T_X Y + T_Y X = 0 \tag{2.3}$$

$$K_{XY} Z + K_{YX} Z = 0 \tag{2.4}$$

$$\langle K_{XY} Z, W \rangle + \langle K_{WZ} X, Y \rangle = 0 \tag{2.5}$$

$$K_{XY} \cdot T = 0 \tag{2.6}$$

$$K_{XY} \cdot K = 0 \tag{2.7}$$

$$\underset{XYZ}{\mathfrak{S}} (K_{XY} Z + T_{T_X Y} Z) = 0 \tag{2.8}$$

$$\underset{XYZ}{\mathfrak{S}} K_{T_X Y Z} = 0, \tag{2.9}$$

where K_{XY} is acting as a derivation on the tensor algebra of V. When a tensor P defining a geometric structure on V is present, we say that (K, T, P) is an infinitesimal model if (K, T) is an infinitesimal model and moreover

$$K_{XY} \cdot P = 0.$$

Let $\widetilde{\nabla}$ be an AS-connection on (M, g) with associated homogeneous structure S. Fixing a point $x \in M$, Ambrose-Singer equations assure that $\widetilde{\nabla}$ determines an infinitesimal model by setting $V = T_xM$, and

$$\begin{aligned} T_X Y &= (S_x)_Y X - (S_x)_X Y, \\ K_{XY} &= (\widetilde{R}_x)_{XY}. \end{aligned}$$

If a geometric structure defined by a tensor field P is present in (M, g), then one takes P_p, and thus $\widetilde{\nabla} P = 0$ implies $K_{XY} \cdot P_p = 0$. Now, from every infinitesimal model (K, T) one can construct a Lie algebra via the so called *Nomizu construction*:

$$\mathfrak{g}_0 = V \oplus \mathfrak{h}_0,$$

where

$$\mathfrak{h}_0 = \{A \in \mathfrak{so}(V) / \, A \cdot K = 0, \, A \cdot T = 0\}.$$

Defining the brackets

$$\begin{aligned} &[A, B] = AB - BA, \qquad A, B \in \mathfrak{k}, \\ &[A, X] = A \cdot X, \qquad A \in \mathfrak{k}, X \in V, \\ &[X, Y] = -T_X Y + K_{XY}, \qquad X, Y \in V, \end{aligned}$$

conditions (2.3) to (2.9) thus imply that $\mathfrak{g}_0$ has a Lie algebra structure. When the geometric structure P is taken into account, one has to take

$$\mathfrak{h}_0 = \{A \in \mathfrak{so}(V) / \, A \cdot K = 0, \, A \cdot T = 0, \, A \cdot P = 0\}.$$

Obtaining Nomizu's construction $\mathfrak{g}_0$ from the infinitesimal model (K, T) is not always an easy task, since the computations required to find $\mathfrak{h}_0$ can be really involved. As an alternative one can consider the so called *transvection algebra* (see [39]). This algebra is defined as $\mathfrak{g}'_0 = V \oplus \mathfrak{h}'_0$, where $\mathfrak{h}'_0$ is the Lie algebra generated by the endomorphisms K_{XY} for all $X, Y \in V$ (the same definition is valid when a geometric structure is present). In general $\mathfrak{g}'_0$ is a proper subalgebra of $\mathfrak{g}_0$. When (K, T) is the infinitesimal model associated to an AS-connection $\widetilde{\nabla}$, then $\mathfrak{h}'_0$ coincides with the holonomy algebra of $\widetilde{\nabla}$.

We now consider the abstract simply-connected Lie group G_0 with Lie algebra $\mathfrak{g}_0$, and its connected Lie subgroup H_0 with Lie algebra $\mathfrak{h}_0$. We also consider the simply-connected Lie group G'_0 with Lie algebra $\mathfrak{g}'_0$, and its connected Lie subgroup H'_0 with Lie algebra $\mathfrak{h}'_0$

Definition 2.3.3 *We say that the infinitesimal model (K, T) (or (K, T, P)) is regular if H_0 is closed in G_0. On the other hand, we say that the transvection algebra $(\mathfrak{g}'_0, \mathfrak{h}'_0)$ is regular if H'_0 is closed in G'_0.*

In the case when the infinitesimal model (resp. the transvection algebra) is regular, one can take the homogeneous space G_0/H_0 (resp. G'_0/H'_0), which will be called the associated *homogenous model* (or the homogeous model associated to $\widetilde{\nabla}$ or S if the infinitesimal model or the transvection algebra come from an AS-connection $\widetilde{\nabla}$ with homogeneous structure S). It is worth noting that following the proof of Ambrose-Singer Theorem (or Kiričenko's Theorem), starting from a homogeneous pseudo-Riemannian manifold (M, g), every AS-connection (or ASK-connection) $\widetilde{\nabla}$ gives a Lie group $\bar{G} = \widetilde{\mathcal{P}}(u_0)$ acting transitively by isometries on (M, g), where $\widetilde{P}(u_0)$ is the holonomy bundle of $\widetilde{\nabla}$ through u_0. Its isotropy subgroup $\bar{H}$ is closed in $\bar{G}$, so that M is diffeomorphic to $\bar{G}/\bar{H}$. The Lie group G'_0 is then nothing but the universal cover of $\bar{G}$, so that H'_0 is closed in G'_0, and M is diffeomorphic to G'_0/H'_0.

An interesting feature about these constructions is that different AS-connections on (M,g) might give different representations of M as a coset G/H. To understand this phenomenon we need the following results and definitions.

Definition 2.3.4 *Two homogeneous structures S and S' on pseudo-Riemannian manifolds (M,g) and (M',g') are said to be isomorphic if there exists an isometry $\varphi : M \to M'$ such that*

$$\varphi_* S' = S.$$

Note that an isomorphism φ between two homogeneous structures S and S' is an affine transformation between $\widetilde{\nabla} = \nabla - S$ and $\widetilde{\nabla}' = \nabla' - S'$. Let now $\mathfrak{g}_0 = V \oplus \mathfrak{h}_0$ and $\mathfrak{g}_0' = V' \oplus \mathfrak{h}_0'$ be the Nomizu constructions associated to S and S' respectively.

Theorem 2.3.5 (see [60]) *If S and S' are isomorphic, then there is a Lie algebra isomorphism $\psi : \mathfrak{g}_0 \to \mathfrak{g}_0'$ such that $\psi(V) = V'$ and $\psi(\mathfrak{h}_0) = \mathfrak{h}_0'$. Moreover, the restriction of ψ to V is an isometry with respect to the scalar products inherited by V and V' from g_0 and g_0' respectively.*

Proof. Let φ be an isomorphism between S and S'. We define $\psi_{|V} = \varphi_*$ and $\psi(A) = \varphi^* A$ for $A \in \mathfrak{h}_0$. As a straightforward computation shows ψ satisfies the statement. ∎

Under suitable topological conditions we have the converse result:

Theorem 2.3.6 *Let (M,g) and (M',g') be connected, simply-connected and complete manifolds with homogeneous structures S and S' respectively. If there exists a Lie algebra isomorphism $\psi : \mathfrak{g}_0 \to \mathfrak{g}_0'$ such that $\psi(V) = V'$, $\psi(\mathfrak{h}_0) = \mathfrak{h}_0'$, and $\psi_{|V}$ is an isometry, then S and S' are isomorphic.*

Proof. As ψ is a Lie algebra isomorphism we have that $\psi_{|V}$ is an isometry between $V = T_xM$ and $V' = T_{x'}M'$ preserving the curvature and torsion tensor fields of $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ respectively. This implies that there are neighborhoods $\mathcal{U}$ and $\mathcal{U}'$ of x and x' respectively, and an affine transformation of $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ $\varphi : \mathcal{U} \to \mathcal{U}'$ taking x to x' and whose differential at x coincides with $\psi_{|V}$ (see [38, Vol. I, Ch. VI]). Since $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ are metric connections and the differential of φ at x is an isometry, φ is an isometry. In addition, since (M,g) and (M',g') are connected, simply-connected and complete, φ can be extended to a global isometry (see again [38, Vol. I, Ch. VI]). ∎

Remark 2.3.7 *Under the hypotheses of the previous Theorem we conclude that (M,g) and (M',g') are homogeneous pseudo-Riemannian manifolds whose simply-connected isometry groups G_0 and G_0' constructed from S and S' respectively are isomorphic. Their corresponding isotropy groups H_0 and H_0' are also isomorphic.*

It is worth noting that given a homogeneous structure S on (M,g) and an isometry $\varphi : M \to M$. The tensor field $S' = \varphi^* S$ is also a homogeneous structure on M, which is in general distinct from S. On the other hand there are examples, for instance in the Heisenberg group (see [60]) such that $\varphi^* S = S$ for every isometry φ. This exhibits that in general there is not uniqueness of solutions of Ambrose-Singer equations, and the existence of isomorphic homogeneous structures does not completely explain the existence of multiple solutions. Actually, we have two different situations:

(i) There exist two non-isomorphic homogeneous structures S_1 and S_2 on (M,g) giving rise to the same Lie algebra $\mathfrak{g}_0$ but with two different decompositions

$$\mathfrak{g}_0 = V_1 \oplus \mathfrak{h}_1 = V_2 \oplus \mathfrak{h}_2,$$

this meaning that there is no isomorphism $\mathfrak{g}_0 \to \mathfrak{g}_0$ with $\psi(V_1) = V_2$, $\psi(\mathfrak{h}_1) = \mathfrak{h}_2$ and $\psi_{|V_1}$ an isometry. This is the case of the Heisenberg group.

(ii) There exist two homogeneous structures S_1 and S_2 on (M, g) with non-isomorphic Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. This implies that there are two different representations of M as a coset, namely G_1/H_1 and G_2/H_2. An example of this situation can be found for instance in the standard 6-dimensional Riemannian sphere:

$$S^6 = SO(7)/SO(6) = G_2/SU_3.$$

Chapter 3

Locally homogeneous pseudo-Riemannian manifolds

Let (M, g) be a Riemannian manifold, then we have the following result (see for instance [59]).

Theorem 3.0.8 *(M, g) is locally homogeneous if and only if it admits an AS-connection.*

This Theorem is no longer true if we replace g by a metric with signature. As a counter example one can take a non-reductive globally homogeneous pseudo-Riemannian manifold. All the proofs of Theorem 3.0.8 known by the author use the "canonical" AS-connection constructed by Kowalski [40]. The construction of this AS-connection relies strongly on the fact that the Killing form of $\mathfrak{so}(T_pM)$ is definite if the metric g is definite. Nevertheless, in [31] it was proved that if a globally homogeneous pseudo-Riemannian manifold $(G/H, g)$ admits an AS-connection, then G/H is reductive. This suggests that to obtain Theorem 3.0.8 in the pseudo-Riemannian case, one has to add a condition playing the same role as the reductivity condition for globally homogeneous spaces.

The aim of this chapter is to formulate and prove an analogous result to Theorem 3.0.8 for pseudo-Riemannian manifolds. We shall also extend that Theorem to the case when an invariant geometric structure is present. In addition, in Section 3.2 we show, under suitable conditions, how to adapt the construction of the "canonical" AS-connection made by Kowalski to metrics with signature. As a consequence we will see that, under those suitable conditions, a locally homogeneous pseudo-Riemannian manifold can be recovered from the curvature and their covariant derivatives at some point up to finite order (see [49] for the Riemannian case). An analogous result will hold in the presence of an invariant geometric structure.

3.1 Reductive locally homogeneous pseudo-Riemannian manifolds

For a comprehensive introduction on Lie pseudo-groups and transitive Lie algebras see [57] and the references therein. We just recall that a transitive Lie algebra is a pair (L, L^0), where L is a Lie algebra and L^0 is a proper subalgebra such that the only ideal of L contained in L^0 is $\{0\}$.

Let (M, g) be a locally homogeneous pseudo-Riemannian space. We denote by $\mathcal{I}$ the pseudo-group of all local isometries, which acts transitively on (M, g). All the elements of $\mathcal{I}$ satisfy the system of PDE's

$$f^*g = g,$$

turning $\mathcal{I}$ into a Lie pseudo-group. The corresponding system of Lie equations is thus

$$\mathcal{L}_X g = 0, \tag{3.1}$$

that is, infinitesimal transformations are given by local Killing vector fields. Let $p \in M$ be fixed, we take $V = T_pM$ and choose a basis $\{e_1, \dots, e_m\}$. The set $\{e^1, \dots, e^m\}$ denotes its dual basis. We consider the transitive Lie algebra $(\mathfrak{i}, \mathfrak{i}^0)$ associated to the system (3.1). The Lie algebra $\mathfrak{i}$ is the set of vector valued formal power series

$$\xi = \sum_{r,i,j_1,\dots,j_r} \xi^i_{j_1\dots j_r} e_i \otimes e^{j_1} \odot \dots \odot e^{j_r},$$

such that $\xi^i_{j_1\dots j_r}$ solve 3.1 and all its derivatives. The subalgebra $\mathfrak{i}^0$ is formed by all the elements of $\mathfrak{i}$ such that the terms of order zero ξ^i vanish. As seen in [57], an element $\xi \in \mathfrak{i}$ is completely determined by the terms of order 0 and 1, which lie in V and $\mathfrak{so}(V)$ respectively.

Definition 3.1.1 *A Killing generator at p is a pair $(X, A) \in T_pM \times \mathfrak{so}(T_pM)$ verifying*

$$A \cdot \nabla^i R_p + i_X \nabla^{i+1} R_p = 0, \qquad i \geq 0,$$

where ∇ is the Levi-Civita connection of g.

The set $\mathfrak{kill}$ of Killing generators at p has a Lie algebra structure with bracket

$$[(X, A), (Y, B)] = (AX - BY, (R_p)_{XY} + [A, B]).$$

We define

$$\mathfrak{kill}^0 = \{(X, A) \in \mathfrak{kill} / \, X = 0\}.$$

Lemma 3.1.2 *[57] $(\mathfrak{kill}, \mathfrak{kill}^0)$ is a transitive Lie algebra isomorphic to $(\mathfrak{i}, \mathfrak{i}^0)$.*

Proof. Let $(x^1, \dots, x^m)$ be a set of normal coordinates around p. We consider the map

$$\begin{array}{ccc} \mathfrak{i} & \to & \mathfrak{kill} \\ (\xi^i, \xi^i_j) & \mapsto & (\xi^i \partial_{x^i|p}, \xi^i_j \partial_{x^i|p} \otimes dx^j_{|p}), \end{array}$$

where (ξ^i, ξ^i_j) are the terms of order 0 and 1 characterizing an element $\xi \in \mathfrak{i}$. A straightforward computation shows that this defines a Lie algebra isomorphism. ■

Let now ξ be a local vector field on M, we define the $(1,1)$-tensor field

$$A_\xi = \mathcal{L}_\xi - \nabla_\xi = -\nabla\xi.$$

Among the equations that ξ must satisfy at p, we have

$$(\mathcal{L}_\xi g)_p = 0,$$

$$(\mathcal{L}_\xi \nabla^i R)_p = 0, \qquad i \geq 0,$$

which coincide with

$$A \cdot g_p = 0,$$

$$A \cdot \nabla^i R_p + i_X \nabla^{i+1} R_p = 0, \qquad i \geq 0,$$

for $X = \xi_p$ and $A = A_\xi|_p$, whence $(\xi_p, A_\xi|_p)$ is a Killing generator.

Corollary 3.1.3 *Every formal solution $\xi \in \mathfrak{i}$ is realized by the germ of a local Killing vector field.*

Proof. Adapting the arguments used by Nomizu in [50] to metrics with signature we can see that if the dimension of the Lie algebra of Killing generators is constant on M, then for every Killing generator (X, A) at a point p there exist a local Killing vector field ξ with $(X, A) = (\xi_p, A_\xi|_p)$. ■

The Lie algebra isomorphism exhibited in the proof of Lemma 3.1.2 can be seen as

$$\begin{array}{ccc} \mathfrak{i} & \to & \mathfrak{kill} \\ [\xi] & \mapsto & (\xi_p, A_\xi|_p), \end{array}$$

where $[\xi]$ denotes the germ of the local vector field ξ at p.

We now consider a Lie pseudo-group $\mathcal{G} \subset \mathcal{I}$ acting transitively on (M, g). A Lie subalgebra $\mathfrak{g} \subset \mathfrak{i}$ can be attached to $\mathcal{G}$, namely, the set of germs of local Killing vector fields with 1-parameter group contained in $\mathcal{G}$. The Lie algebra $\mathfrak{k}$ formed by those $[\xi] \in \mathfrak{g}$ vanishing at p is thus a Lie subalgebra of $\mathfrak{i}^0$, and the pair $(\mathfrak{g}, \mathfrak{k})$ is a transitive Lie algebra.

Definition 3.1.4 *Let $\mathcal{G}$ be a Lie pseudo-group acting transitively on (M, g). The isotropy pseudo-group at a point $p \in M$ is*

$$\mathcal{H}_p = \{f \in \mathcal{G} / \, f(p) = p\} \subset \mathcal{G}.$$

Since $f(p) = p$ is not a differential equation, $\mathcal{H}_p$ is not a Lie pseudo-group in general. For this reason it is more convenient to work with the linear isotropy group.

Definition 3.1.5 *The linear isotropy group of $\mathcal{G}$ at $p \in M$ is*

$$H_p = \{F : T_pM \to T_pM / \, F = f_*, \, f \in \mathcal{H}_p\}.$$

Since every $f \in \mathcal{H}_p$ is an isometry, H_p is a Lie subgroup of $O(T_pM)$.

Lemma 3.1.6 *The Lie algebra $\mathfrak{h}_p$ of H_p is isomorphic to $\mathfrak{k}$.*

Proof. We define the map

$$\begin{array}{ccc} \mathfrak{k} & \to & \mathfrak{h}_p \\ [\xi] & \mapsto & \frac{d}{dt}\big|_{t=0} (f_t)_*, \end{array}$$

where $f_t \subset \mathcal{H}_p$ is the 1-parameter group generated by ξ. A simple inspection shows that this map is a Lie algebra isomorphism. ■

Note that the previous isomorphism between $\mathfrak{k}$ and $\mathfrak{h}_p$ can be read as

$$\begin{array}{ccc} \mathfrak{k} & \to & \mathfrak{h}_p \\ [\xi] & \mapsto & A_\xi|_p. \end{array}$$

There is a natural action of H_p on $\mathfrak{g}$ given by

$$\mathrm{Ad} : \begin{array}{ccc} H_p \times \mathfrak{g} & \to & \mathfrak{g} \\ (F, [\xi]) & \mapsto & [\eta] \,, \end{array}$$

with

$$\eta_q = \frac{d}{dt}\bigg|_{t=0} f \circ \varphi_t \circ f^{-1}(q),$$

for every q in a neighborhood of p, where φ_t is the 1-parameter group generated by $[\xi]$, and $F = f_*$. When identifying $\mathfrak{k}$ with $\mathfrak{h}_p$ the restriction of this action to $\mathfrak{k}$ is just the usual adjoint action of H_p on its Lie algebra, so the notation is consistent.

Definition 3.1.7 *Let (M,g) be a pseudo-Riemannian manifold, and let $\mathcal{G}$ be a Lie pseudo-group of isometries acting transitively on (M,g). We will say that the triple $(M,g,\mathcal{G})$ is reductive if the transitive Lie algebra $(\mathfrak{g},\mathfrak{k})$ associated to $\mathcal{G}$ can be decomposed as $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{k}$, where $\mathfrak{m}$ is $\mathrm{Ad}(H_p)$-invariant.*

Note that being reductive is a property of the triple $(M,g,\mathcal{G})$ rather than a property of the pseudo-Riemannian manifold (M,g) itself. In Section 3.4 we will show that a same locally homogeneous pseudo-Riemannian manifold can be reductive for the action of certain Lie pseudo-group $\mathcal{G}$, whereas it is non-reductive for the action of another Lie pseudo-group $\mathcal{G}'$. On the other hand, it seems that the previous definition depends on the chosen point $p\in M$, however

Proposition 3.1.8 *If $(M,g,\mathcal{G})$ is reductive at a point $p\in M$, then it is reductive at every point $q\in M$.*

Proof. Let q be another point in M. We denote by $(\mathfrak{g}_p,\mathfrak{k}_p)$ and $(\mathfrak{g}_q,\mathfrak{k}_q)$ the transitive Lie algebras associated to $\mathcal{G}$ at p and q respectively. Let $h\in\mathcal{G}$ be a local isometry with $h(p)=q$. h induces isomorphisms $\hat{h}:\mathfrak{g}_p\to\mathfrak{g}_q$, $[\xi]\mapsto[h_*(\xi)]$, and $\check{h}:H_p\to H_q$, $F\mapsto h_*\circ F\circ h_*^{-1}$. Let $\mathfrak{g}_p=\mathfrak{m}_p\oplus\mathfrak{k}_p$ with $\mathfrak{m}_p$ $\mathrm{Ad}(H_p)$-invariant, we define $\mathfrak{m}_q=\hat{h}(\mathfrak{m}_p)\subset\mathfrak{g}_q$. It is obvious that $\mathfrak{g}_q=\mathfrak{m}_q\oplus\mathfrak{k}_q$, since $\hat{h}$ is an isomorphism and takes $\mathfrak{k}_p$ to $\mathfrak{k}_q$. We now show that $\mathfrak{m}_q$ is $\mathrm{Ad}(H_q)$-invariant and independent of the local isometry h. Let $F\in H_q$, and let $f\in\mathcal{H}_q$ with $F=f_*$. Let $[\eta]\in\mathfrak{m}_q$, there is an element $[\xi]\in\mathfrak{m}_p$ with $\eta=h_*(\xi)$. The 1-parameter group generated by η is thus $\phi_t=h\circ\varphi_t\circ h^{-1}$, where φ_t is the 1-parameter group generated by ξ. Therefore

$$
\begin{aligned}
\mathrm{Ad}_F([\eta]) &= \left[\left.\frac{d}{dt}\right|_{t=0} f\circ\phi_t\circ f^{-1}\right]\\
&= \left[\left.\frac{d}{dt}\right|_{t=0} f\circ h\circ\varphi_t\circ h^{-1}\circ f^{-1}\right]\\
&= \left[\left.\frac{d}{dt}\right|_{t=0} h\circ h^{-1}\circ f\circ h\circ\varphi_t\circ h^{-1}\circ f^{-1}\circ h\circ h^{-1}\right]\\
&= h_*\left(\mathrm{Ad}_{\check{h}^{-1}(F)}([\xi])\right).
\end{aligned}
$$

Since $\check{h}^{-1}(F)\in H_p$, we have $\mathrm{Ad}_F([\eta])\in\mathfrak{m}_q$. On the other hand, in order to prove the independence of h, it is enough to prove that for other $h'\in\mathcal{G}$ with $h'(p)=q$ we have that $h_*^{-1}\circ h'_*([\xi])\in\mathfrak{m}_p$. But

$$
h_*^{-1}\circ h'_*([\xi]) = \left[\left.\frac{d}{dt}\right|_{t=0} h^{-1}\circ h'\circ\varphi_t\right] = \mathrm{Ad}_{(h^{-1}\circ h')_*}([\xi]).
$$

Since $h^{-1}\circ h'\in\mathcal{H}_p$ and $\mathfrak{m}_p$ is $\mathrm{Ad}(H_p)$-invariant we conclude that $h_*^{-1}\circ h'_*([\xi])\in\mathfrak{m}_p$. ∎

The following two theorems characterize locally homogeneous pseudo-Riemannian manifolds admitting an AS-connection (see Definition 2.2.8).

Theorem 3.1.9 *Let $(M,g,\mathcal{G})$ be a reductive locally homogeneous pseudo-Riemannian manifold. Then (M,g) admits an AS-connection.*

Proof. Let (r,s) be the signature of g, and let $\mathcal{O}(M)$ be the bundle of orthonormal references of M. We fix a point $p\in M$ and and a reference $u_0\in\mathcal{O}(M)$ in the fiber of p. We shall interpret an orthonormal reference u at $q\in M$ as an isometry $u:(\mathbb{R}^m,\langle\,,\,\rangle)\to(T_qM,g_q)$, where $\langle\,,\,\rangle$ is the standard metric of $\mathbb{R}^m$ with signature (r,s). Consider the set

$$
Q=\{u\in\mathcal{O}(M)/\,u=\widetilde{h}(u_0),\ h\in\mathcal{G}\}, \tag{3.2}
$$

where $\widetilde{h}$ is the map induced on $\mathcal{O}(M)$ by a local isometry h. Q determines a reduction of $\mathcal{O}(M)$ with structure group

$$\bar{H} = \{B \in O(r,s)/\, \hat{u}_0(B) = f_*,\, f \in \mathcal{H}_p\},$$

where $\hat{u}_0 : O(r,s) \to O(T_pM)$, $B \mapsto u_0 \circ B \circ u_0^{-1}$. It is obvious that $\hat{u}_0$ gives an isomorphism between $\bar{H}$ and the linear isotropy group H_p. The right action of an element $B \in \bar{H}$ on a reference $u \in Q$ at q is given by $R_B(u) = u \circ B : \mathbb{R}^m \to \mathbb{R}^m \to T_qM$. Let $F = \hat{u}_0(B) \in H_p$ and $f \in \mathcal{H}_p$ with $F = f_*$. Let $h \in \mathcal{G}$ be such that $u = \widetilde{h}(u_0)$, we can write

$$\begin{aligned} R_B(u) &= u \circ B = u \circ u_0 \circ F \circ u_0 \\ &= h_* \circ u_0 \circ u_0^{-1} \circ F \circ u_0 = h_* \circ f_* \circ u_0 \\ &= \widetilde{h} \circ \widetilde{f}(u_0). \end{aligned}$$

We now consider the map

$$\begin{array}{rccl} \Psi : & \mathfrak{g} & \to & T_{u_0}Q \\ & [\xi] & \to & \widetilde{\xi}_{u_0} = \frac{d}{dt}\big|_{t=0} \widetilde{\varphi_t}(u_0), \end{array}$$

where φ_t is the 1-parameter group of ξ. Recall that $\widetilde{\xi}$ is the natural lift of ξ as defined in Lemma 1.3.9. Ψ is injective as $\{\varphi_t\} \subset \mathcal{G}$ and the action of $\mathcal{G}$ on Q is free. Moreover,

$$\dim\mathfrak{g} = \dim T_pM + \dim\mathfrak{k} = \dim T_pM + \dim V_{u_0}Q = \dim T_{u_0}Q,$$

whence Ψ is a linear isomorphism. Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ be a reductive decomposition, we define the horizontal subspace at u_0 as

$$H_{u_0}Q = \Psi(\mathfrak{m}),$$

and making use of $\mathcal{G}$ we define an horizontal distribution on Q as

$$H_uQ = \widetilde{h}_*(H_{u_0}), \qquad u = \widetilde{h}(u_0).$$

This horizontal distribution is $\mathcal{C}^\infty$ and invariant by $\mathcal{G}$. In order to see that HQ defines a linear connection $\widetilde{\nabla}$ on M we just have to show that it is equivariant by the right action of $\bar{H}$. Let $B \in \bar{H}$, we take $F = \hat{u}_0(B)$, and $f \in \mathcal{H}_p$ with $F = f_*$. Let $X_u \in H_uQ$, by definition $X_u = \widetilde{h}_*(X_{u_0})$ for some $X_{u_0} \in H_{u_0}Q$ and some h such that $u = \widetilde{h}(u_0)$. This means that $X_u = \widetilde{h}_*(\Psi([\xi]))$ for some $[\xi] \in \mathfrak{m}$. Let φ_t be the 1-parameter group generated by ξ, we thus have

$$\begin{aligned} (R_B)_*(X_u) &= (R_B)_* \circ \widetilde{h}_* \circ \Psi([\xi]) \\ &= \frac{d}{dt}\bigg|_{t=0} R_B \circ \widetilde{h} \circ \widetilde{\varphi_t}(u_0) = \frac{d}{dt}\bigg|_{t=0} \widetilde{h} \circ \widetilde{\varphi_t} \circ \widetilde{f}(u_0) \\ &= \frac{d}{dt}\bigg|_{t=0} \widetilde{h} \circ \widetilde{f} \circ \widetilde{f}^{-1} \circ \widetilde{\varphi_t} \circ \widetilde{f}(u_0) \\ &= (\widetilde{h} \circ \widetilde{f})_* \left(\frac{d}{dt}\bigg|_{t=0} \widetilde{f}^{-1} \circ \widetilde{\varphi_t} \circ \widetilde{f}(u_0) \right) \\ &= (\widetilde{h} \circ \widetilde{f})_* \left(\Psi(\mathrm{Ad}_{F^{-1}}([\xi])) \right). \end{aligned}$$

Since $\mathrm{Ad}_{F^{-1}}([\xi]) \in \mathfrak{m}$, we have $\Psi(\mathrm{Ad}_{F^{-1}}([\xi])) \in H_{u_0}Q$, whence $(R_B)_*(X_u) \in H_{R_B(u)}Q$ since $R_B(u) = \widetilde{h} \circ \widetilde{f}(u_0)$.

We now study the properties of the connection $\widetilde{\nabla}$. Firstly, since Q is a reduction of $\mathcal{O}(M)$, the connection $\widetilde{\nabla}$ is metric, that is, $\widetilde{\nabla} g = 0$. On the other hand, the connection

$\widetilde{\nabla}$ is characterized in the following way. Let $p, q \in M$, and let γ be a path in M with $\gamma(0) = p$ and $\gamma(1) = q$. We denote by $\bar{\gamma}$ the horizontal lift of γ to $u_0 \in Q$ with respect to $\widetilde{\nabla}$. The parallel transport along γ with respect to this connection is thus the linear isometry $\gamma : T_pM \to T_qM$ given by $\gamma = u \circ u_0^{-1}$, where $u = \bar{\gamma}(1)$. But since $u = \widetilde{h}(u_0) = h_* \circ u_0$ for some $h \in \mathcal{G}$, we have that the linear isometry γ is exactly h_*. This characterization of $\widetilde{\nabla}$ implies that its torsion $\widetilde{T}$ and curvature $\widetilde{R}$ are invariant by parallel transport, since $\widetilde{\nabla}$ is invariant by $\mathcal{G}$, that is $\widetilde{\nabla}\widetilde{T} = 0$ and $\widetilde{\nabla}\widetilde{R} = 0$. As usual, this two equations are equivalent to

$$\widetilde{\nabla}R = 0, \qquad \widetilde{\nabla}S = 0,$$

where R is the curvature of g, and $S = \nabla - \widetilde{\nabla}$ with ∇ the Levi-Civita connection of g. This proves that $\widetilde{\nabla}$ is an AS-connection. ∎

Theorem 3.1.10 *Let (M, g) be a pseudo-Riemannian manifold admitting an AS-connection $\widetilde{\nabla}$. Then there is a Lie pseudo-group of isometries $\mathcal{G}$ such that $(M, g, \mathcal{G})$ is reductive locally homogeneous.*

Proof. Let $p, q \in M$, we consider a path γ from p to q. Since $\widetilde{\nabla}$ is an AS-connection, the parallel transport $\gamma : T_pM \to T_qM$ with respect to $\widetilde{\nabla}$ is a linear isometry preserving the torsion and curvature of $\widetilde{\nabla}$. This implies that there exist neighborhoods $\mathcal{U}^p$ and $\mathcal{U}^q$, and an affine transformation $f^\gamma : \mathcal{U}^p \to \mathcal{U}^q$ with respect to $\widetilde{\nabla}$, such that its differential at p coincides with the parallel transport along γ (see [38, Vol. I, Ch. VI]). Since $\widetilde{\nabla}$ is metric we have that f^γ is an isometry. We consider the set

$$\mathcal{G} = \{f^\gamma /\, \gamma \text{ is a path from } p \text{ to } q\}.$$

$\mathcal{G}$ is a pseudo-group of local isometries of (M, g) which acts transitively on (M, g), so that (M, g) is locally homogeneous. In addition, $\mathcal{G}$ coincides with the so called transvection group of $\widetilde{\nabla}$, which consists of all local affine maps of $\widetilde{\nabla}$ preserving its holonomy bundle $\mathcal{P}^{\widetilde{\nabla}}$, that is, $\widetilde{f}(\mathcal{P}^{\widetilde{\nabla}}) \subset \mathcal{P}^{\widetilde{\nabla}}$. This gives $\mathcal{G}$ a structure of Lie pseudo-group. We just have to show that $(M, g, \mathcal{G})$ is reductive. For a fixed point $p \in M$, the isotropy pseudo-group is

$$\mathcal{H}_p = \{f^\gamma /\, f^\gamma(p) = p\} \overset{1:1}{\leftrightarrow} \{\text{loops based at } p\}.$$

The linear isotropy group is thus

$$H_p = \{f^\gamma_* : T_pM \to T_pM /\, f^\gamma \in \mathcal{H}_p\} = Hol^{\widetilde{\nabla}}.$$

Therefore, let $(\mathfrak{g}, \mathfrak{k})$ be the transitive Lie algebra associated to $\mathcal{G}$, we have $\mathfrak{k} \simeq \mathfrak{hol}^{\widetilde{\nabla}}$. Fix an orthonormal reference u_0 at p and consider the bundle Q defined as in (3.2). It is obvious that Q is exactly the holonomy bundle of $\widetilde{\nabla}$ at u_0, and therefore, the connection $\widetilde{\nabla}$ reduces to Q and determines a horizontal distribution HQ which is invariant by the right action of H_p and by the left action of $\mathcal{G}$ on Q (recall that all the elements of $\mathcal{G}$ are affine maps with respect to $\widetilde{\nabla}$). We again take the linear map

$$\begin{array}{rccc} \Psi : & \mathfrak{g} & \to & T_{u_0}Q \\ & [\xi] & \mapsto & \widetilde{\xi}_{u_0}. \end{array}$$

As seen before Ψ is a linear isomorphism. We consider the subspace $\mathfrak{m} = \Psi^{-1}(H_{u_0}Q) \subset \mathfrak{g}$. Obviously $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, as $\Psi(\mathfrak{k}) = V_{u_0}Q$. In addition, let $[\xi] \in \mathfrak{m}$ with 1-parameter group φ_t, and let $F = f_* \in H_p$, recall that $\mathrm{Ad}_F([\xi]) = [\eta]$ with $\eta_q = \frac{d}{dt}\big|_{t=0} f \circ \varphi_t \circ f^{-1}(q)$ for every q in a neighborhood of p. Hence

$$\Psi(\mathrm{Ad}_F([\xi])) = \left.\frac{d}{dt}\right|_{t=0} \widetilde{f} \circ \varphi_t \circ \widetilde{f}^{-1}(u_0) = \widetilde{f}_* \left((R_{F^{-1}})_*(\widetilde{\xi}_{u_0}) \right).$$

Since $[\xi] \in \mathfrak{m}$ we have that $\widetilde{\xi}_{u_0} \in H_{u_0}Q$, whence by the invariance and the equivariance of the horizontal distribution

$$\widetilde{f}_* \left((R_{F^{-1}})_* (\widetilde{\xi}_{u_0}) \right) \in \widetilde{f}_* \left(H_{R_{F^{-1}}(u_0)} Q \right) = H_{u_0}Q.$$

This implies that $\mathfrak{m}$ is $\mathrm{Ad}(H_p)$-invariant, showing that (M, g) is reductive. ∎

Remark 3.1.11 *Obviously a globally homogeneous pseudo-Riemannian manifold is in particular a locally homogeneous pseudo-Riemannian manifold. Therefore the notion of reductivity that we have defined for locally homogeneous pseudo-Riemannian manifolds must coincide with the well known definition of reductive homogeneous spaces when we consider a Lie group G as the Lie pseudo-group $\mathcal{G}$. We show below that this is the case.*

Let (M, g) be a globally homogeneous pseudo-Riemannian manifold with a Lie group G of (global) isometries acting transitively on it. Let H_p be the isotropy group at a point $p \in M$. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. Recall that (M, g, G) is said reductive if $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ for some $\mathrm{Ad}(H_p)$-invariant subspace $\mathfrak{m} \subset \mathfrak{g}$ (see Definition 1.3.7). We denote by $(\mathfrak{g}', \mathfrak{k}')$ the transitive Lie algebra associated to G seen as a Lie pseudo-group of local isometries, i.e., the set of germs of local infinitesimal transformations of G. The linear isotropy group as defined in Definition 3.1.4 is just the image of H_p under the linear isotropy representation λ (see (1.7)). We also recall the definition of fundamental vector fields: let $\alpha \in \mathfrak{g}$ we define the vector field α^* on M as

$$\alpha^*_q = \left.\frac{d}{dt}\right|_{t=0} L_{\exp(t\alpha)}(q), \qquad q \in M,$$

where L_a denotes the left action of $a \in G$ on M. We consider the following map

$$\begin{array}{rccc} \phi: & \mathfrak{g} & \to & \mathfrak{g}' \\ & \alpha & \mapsto & [\alpha^*]\,. \end{array}$$

Note that ϕ is not a Lie algebra homomorphism since $[\alpha, \beta]^* = -[\alpha^*, \beta^*]$. Nevertheless we show that it is a linear isomorphism. Let $\alpha \in \mathfrak{g}$ be such that $[\alpha^*] = 0$, this means that $\alpha^* = 0$ in a neighborhood around p. In particular $\alpha^*_p = 0$ and $A_{\alpha^*}|_p = 0$, so that $\alpha^* = 0$. This implies $\alpha = 0$, that is, ϕ is injective. On the other hand, let $[\xi] \in \mathfrak{g}'$, we consider the 1-parameter group of ξ, which determines a curve $\varphi_t \subset G$. Taking $\alpha = \frac{d}{dt}\big|_{t=0} \varphi_t$ we have $\phi(\alpha) = [\xi]$. This proves that ϕ is surjective. In addition, let $h \in H_p$ so that $h_* \in \lambda(H_p)$, the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi"] \arrow[d, "\mathrm{Ad}_h"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{g}' \arrow[d, "\mathrm{Ad}_{h_*}"] \\ \mathfrak{g} \arrow[r, "\phi"] & \mathfrak{g}' \end{tikzcd}$$

In fact, let $\alpha \in \mathfrak{g}$, then $\mathrm{Ad}_{h_*}(\alpha^*) = [\eta]$ with

$$\eta_q = \left.\frac{d}{dt}\right|_{t=0} L_h \circ L_{\exp t\alpha} \circ L_{h^{-1}} = (L_h)_* \left(\alpha^*_{L_{h^{-1}}(q)} \right) = (\mathrm{Ad}_h(\alpha))^*_q.$$

We conclude that making use of ϕ one can transform reductive complements of $(\mathfrak{g}, \mathfrak{h})$ to reductive complements of $(\mathfrak{g}', \mathfrak{k}')$ and viceversa. This means that the notions of reductivity from both the global and the local points of view coincide.

We finally show a necessary condition for a reductive locally homogeneous pseudo-Riemannian manifold to be locally isometric to a globally homogeneous pseudo-Riemannian manifold. This question has already been solved in the Riemannian case (see for instance [39] and [57]).

Proposition 3.1.12 *Let $(M, g, \mathcal{G})$ be a reductive locally homogeneous pseudo-Riemannian manifold endowed with an associated AS-connection $\widetilde{\nabla}$. If the infinitesimal model (K, T) of $\widetilde{\nabla}$ is regular (see Definition 2.3.3), then (M, g) is locally isometric to a reductive globally homogeneous pseudo-Riemannian manifold. The same holds if the transvection algebra $(\mathfrak{g}_0', \mathfrak{hol}^{\widetilde{\nabla}})$ is regular.*

Proof. Let $p \in M$, consider the Nomizu construction $\mathfrak{g}_0 = T_pM \oplus \mathfrak{h}_0$ associated to (K, T). Let G_0 be the simply-connected Lie group with Lie algebra $\mathfrak{g}_0$, and H_0 its connected subgroup with Lie algebra $\mathfrak{h}_0$. If (K, T) is regular then H_0 is closed in G_0, so that we can consider the homogeneous space G_0/H_0. Moreover, G_0/H_0 is reductive as $\mathfrak{g}_0 = T_pM \oplus \mathfrak{h}_0$ is a reductive decomposition, and the tangent space of G_0/H_0 at the origin o is identified with T_pM through a linear isomorphism $F : T_pM \to T_0(G_0/H_0)$. This homogeneous space is thus endowed with a G_0-invariant pseudo-Riemannian matric inherited from g at p. We consider the canonical connection $\widetilde{\nabla}^{can}$ associated to this reductive decomposition (see Definition 1.3.11). Under the identification F, the curvature and torsion of $\widetilde{\nabla}$ coincides with K and T respectively. This means that there is a linear isometry $F : T_pM \to T_0(G_0/H_0)$ preserving the curvature and torsion of $\widetilde{\nabla}$ and $\widetilde{\nabla}^{can}$. Therefore, there are open neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of p and o, and an affine transformation $f : \mathcal{U} \to \mathcal{V}$ with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^{can}$ taking p to o (see [38, Vol. I, Ch. VI]). Since both connections are metric we have that f is an isometry. The same arguments can be applied substituting the Nomizu construction by the transvection algebra. ■

3.1.1 Locally homogeneous pseudo-Riemannian manifolds with invariant geometric structures

We now consider a locally homogeneous pseudo-Riemannian manifold (M, g) endowed with a geometric structure given by a tensor field P. Recall the definition of an ASK-connection (Definition 2.2.8). Note that an ASK-connection is in particular an AS-connection. We say that the geometric structure given by P is invariant if the Lie pseudo-group of isometries $\mathcal{J}$ preserving P, that is

$$\mathcal{J} = \{f \in \mathcal{I},\ f^*P = P\},$$

acts transitively on M. The corresponding Lie equation is

$$\mathcal{L}_X P,$$

so that the infinitesimal transformations of $\mathcal{G}$ are Killing vector fields which are infinitesimal automorphisms of the geometric structure. A vector field ξ satisfying both $\mathcal{L}_\xi g = 0$ and $\mathcal{L}_\xi P = 0$ will be called a *geometric Killing vector field*. We consider the Lie algebra $\mathfrak{j} \subset \mathfrak{i}$, which consists of germs of geometric Killing vector fields. The Lie subalgebra $\mathfrak{j}^0 \subset \mathfrak{i}^0$ is defined as the set of elements of $\mathfrak{j}$ vanishing at p, so that $(\mathfrak{j}, \mathfrak{j}^0)$ is a transitive Lie algebra. Let $\mathfrak{gkill}$ be the subalgebra of $\mathfrak{kill}$ containing all the Killing generators (X, A) satisfying

$$A \cdot \nabla^j P_p + i_X \nabla^{j+1} P_p = 0, \qquad j \geq 0,$$

and let $\mathfrak{gkill}^0 = \mathfrak{kill}^0 \cap \mathfrak{gkill}$, we have

Proposition 3.1.13 *The transitive Lie algebra $(\mathfrak{gkill}, \mathfrak{gkill}^0)$ is isomorphic to $(\mathfrak{j}, \mathfrak{j}^0)$.*

Proof. Let ξ be a geometric Killing vector field, let $(X, A) = (\xi_p, A_\xi|_p)$. By definition we have

$$A \cdot \nabla^j P = \mathcal{L}_\xi (\nabla^j P)_p - \nabla_\xi \nabla^j P_p = \mathcal{L}_\xi (\nabla^j P)_p - i_X \nabla^{j+1} P_p,$$

and applying Lemma 3.1.14 below we obtain that $(\xi_p, A_\xi|_p) \in \mathfrak{gkill}$. Making use of Lemma 3.1.2 and Corollary 3.1.3 we see that the map

$$\begin{array}{rcl} \mathfrak{j} & \to & \mathfrak{gkill} \\ [\xi] & \mapsto & (\xi_p, A_\xi|_p) \end{array}$$

is a Lie algebra isomorphism taking $\mathfrak{j}^0$ to $\mathfrak{gkill}^0$. ■

Lemma 3.1.14 *Let ξ be a Killing vector field and ω a tensor field. If $\mathcal{L}_\xi\omega = 0$ then $\mathcal{L}_\xi(\nabla\omega) = 0$.*

Proof. For the sake of simplicity we show the proof for ω a 1-form. The generalization for tensor fields of arbitrary type is straightforward. By direct calculation

$$\mathcal{L}_\xi(\nabla\omega)(X,Y) = -\xi\cdot(\omega(\nabla_X Y)) + \omega\left(\nabla_{\mathcal{L}_\xi X} Y\right) + \omega\left(\nabla_X \mathcal{L}_\xi Y\right).$$

Making use of $\mathcal{L}_\xi\omega = 0$ we obtain

$$\mathcal{L}_\xi(\nabla\omega)(X,Y) = \omega\left((\mathcal{L}_\xi\nabla)(X,Y)\right) = \omega\left(R_{\xi X}Y + \nabla^2_{XY}\xi\right).$$

But $R_\xi + \nabla^2\xi = 0$ since it is just the affine Jacobi equation applied to a Killing vector field ξ. ■

We now consider a Lie pseudo-group $\mathcal{G} \subset \mathcal{J}$ acting transitively on M. We associate to $\mathcal{G}$ the Lie algebra $\mathfrak{g} \subset \mathfrak{j}$ consisting on germs of local geometric Killing vector fields with 1-parameter group contained in $\mathcal{G}$. The Lie algebra $\mathfrak{k}$ consisting of those $[\xi] \in \mathfrak{g}$ vanishing at p is thus a Lie subalgebra of $\mathfrak{j}^0$, and the pair $(\mathfrak{g}, \mathfrak{k})$ is a transitive Lie algebra. We take the isotropy pseudo-group $\mathcal{H}_p$ and the linear isotropy group H_p associated to $\mathcal{G}$. As before we have that H_p is a Lie subgroup of the stabilizer of P_p inside $O(T_pM)$ and $\mathfrak{k} \simeq \mathfrak{h}_p$. Recall that we have the action Ad of H_p on $\mathfrak{g}$.

Definition 3.1.15 *Let (M,g,P) be a pseudo-Riemannian manifold endowed with a geometric structure defined by a tensor field P. Let $\mathcal{G}$ be a Lie pseudo-group of isometries acting transitively on (M,g,P) and preserving P. We will say that $(M,g,P,\mathcal{G})$ is reductive if the transitive Lie algebra $(\mathfrak{g},\mathfrak{k})$ associated to $\mathcal{G}$ can be decomposed as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, where $\mathfrak{m}$ is $\mathrm{Ad}(H_p)$-invariant.*

Theorem 3.1.16 *Let $(M,g,P,\mathcal{G})$ be a reductive locally homogeneous pseudo-Riemannian manifold with P invariant. Then (M,g,P) admits an ASK-connection.*

Proof. Let $(M,g,P,\mathcal{G})$ be a reductive locally homogeneous pseudo-Riemannian manifold with P invariant, by Theorem 3.1.9 (M,g) admits an AS-connection $\widetilde{\nabla}$. We just have to show that $\widetilde{\nabla}P = 0$. However, recall that $\widetilde{\nabla}$ is characterized as the linear connection whose parallel transport coincides with the differential h_* for some $h \in \mathcal{G}$. Since $\mathcal{G}$ preserves P, we have that P is invariant by parallel transport with respect to $\widetilde{\nabla}$, whence $\widetilde{\nabla}P = 0$. ■

Theorem 3.1.17 *Let (M,g,P) be a pseudo-Riemannian manifold admitting a ASK-connection $\widetilde{\nabla}$. Then there is a Lie pseudo-group of isometries $\mathcal{G}$ acting transitively on (M,g,P) and preserving P, such that $(M,g,P,\mathcal{G})$ is reductive locally homogeneous with P invariant.*

Proof. As in the proof of Theorem 3.1.10 we consider the Lie pseudo-group

$$\mathcal{G} = \{f^\gamma /\ \gamma \text{ is a path from } p \text{ to } q\}.$$

Since the local maps f^γ are affine maps of $\widetilde{\nabla}$, and $\widetilde{\nabla}P = 0$, we have that P is invariant by $\mathcal{G}$. The same exact arguments used in the proof of Theorem 3.1.10 show that (M,g,P) is reductive locally homogeneous with P invariant. ■

Remark 3.1.18 *Propositions 3.1.12 can be adapted in a straightforward way to the case when (M,g) is endowed with an invariant geometric structure.*

3.2 Strongly reductive locally homogeneous pseudo-Riemannian manifolds

The results we present in this section apply to pseudo-Riemannian metrics of any signature (including the Riemannian case) with or without an extra geometric structure. These results are new for pseudo-Riemannian metrics with signature with or without an extra geometric structure, and in the Riemannian case the results are new in the presence of a geometric structure. For the already known case of Riemannian metrics without extra geometry see [49]. For the sake of brevity we present here the most general case.

Let (M, g) be a pseudo-Riemannian manifold endowed with a geometric structure defined by a tensor field P. Let ∇ and R denote the Levi-Civita connection of g and its curvature tensor field. Let $p \in M$, for every integers $r, s \geq 0$ we consider the Lie algebras $\mathfrak{g}(p,r)$ and $\mathfrak{p}(p,s)$ given by

$$\mathfrak{g}(p,r) = \left\{A \in \mathfrak{so}(T_pM),\ A \cdot \left(\nabla^i R_p\right) = 0,\ \ i = 0, \ldots, r\right\},$$

$$\mathfrak{p}(p,s) = \left\{A \in \mathfrak{so}(T_pM),\ A \cdot \left(\nabla^j P_p\right) = 0,\ \ j = 0, \ldots, s\right\},$$

where A acts as a derivation on the tensor algebra of T_pM. We thus have filtrations

$$\mathfrak{so}(T_pM) \supset \mathfrak{g}(p,0) \supset \ldots \supset \mathfrak{g}(p,r) \supset \ldots$$

$$\mathfrak{so}(T_pM) \supset \mathfrak{p}(p,0) \supset \ldots \supset \mathfrak{p}(p,s) \supset \ldots$$

Let $k(p)$ and $l(p)$ be the first integers such that $\mathfrak{g}(p,k(p)) = \mathfrak{g}(p,k(p)+1)$ and $\mathfrak{p}(p,l(p)) = \mathfrak{p}(p,l(p)+1)$, and let $\mathfrak{h}(p,r,s) = \mathfrak{g}(p,r) \cap \mathfrak{p}(p,s)$. We consider the complex of filtrations

$$\begin{array}{ccccccccc}
\mathfrak{so}(T_pM) & \supset & \mathfrak{g}(p,0) & \supset & \ldots & \supset & \mathfrak{g}(p,k(p)) & = & \mathfrak{g}(p,k(p)+1) \\
\cup & & \cup & & & & \cup & & \cup \\
\mathfrak{p}(p,0) & \supset & \mathfrak{h}(p,0,0) & \supset & \ldots & \supset & \mathfrak{h}(p,k(p),0) & = & \mathfrak{h}(p,k(p)+1,0) \\
\cup & & \cup & & & & \cup & & \cup \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\cup & & \cup & & & & \cup & & \cup \\
\mathfrak{p}(p,l(p)) & \supset & \mathfrak{h}(p,0,l(p)) & \supset & \ldots & \supset & \mathfrak{h}(p,k(p),l(p)) & = & \mathfrak{h}(p,k(p)+1,l(p)) \\
|| & & || & & & & || & & || \\
\mathfrak{p}(p,l(p)+1) & \supset & \mathfrak{h}(p,0,l(p)+1) & \supset & \ldots & \supset & \mathfrak{h}(p,k(p),l(p)+1) & = & \mathfrak{h}(p,k(p)+1,l(p)+1).
\end{array}$$

To complete the notation we will denote $\mathfrak{g}(p,-1) = \mathfrak{so}(T_pM)$, $\mathfrak{p}(p,-1) = \mathfrak{so}(T_pM)$, so that $\mathfrak{h}(p,-1,s) = \mathfrak{p}(p,s)$ and $\mathfrak{h}(p,r,-1) = \mathfrak{g}(p,r)$.

We shall call a pair of integers $(r(p), s(p))$ in the set $\mathbb{N} \cup \{0,-1\}$ a *stabilizing pair* at $p \in M$ if $r(p) \leq k(p)$, $s(p) \leq l(p)$ and

$$\begin{array}{ccc}
\mathfrak{h}(p,r(p),s(p)) & = & \mathfrak{h}(p,r(p)+1,s(p)) \\
|| & & || \\
\mathfrak{h}(p,r(p),s(p)+1) & = & \mathfrak{h}(p,r(p)+1,s(p)+1).
\end{array}$$

Note that $(k(p), l(p))$ is a stabilizing pair.

Remark 3.2.1 *In Section 3.4 we exhibit an example of a manifold with an stabilizing pair distinct from* $(k(p), l(p))$.

The following definition generalizes the definition of infinitesimal homogeneous space given by Singer ([49]). Consider a pair of integers $(r,s) \in (\mathbb{N} \cup \{0,-1\})^2$. We say that

(M,g,P) is (r,s)-*infinitesimally P-homogeneous* if for every $p,q \in M$ there is a linear isometry $F : T_pM \to T_qM$ such that

$$F^*(\nabla^i R_q) = \nabla^i R_p, \qquad i = 0,\dots,r+1,$$
$$F^*(\nabla^j P_q) = \nabla^j P_p, \qquad j = 0,\dots,s+1.$$

Let $p \in M$ be a fixed point and suppose that $(r(p),s(p))$ is a stabilizing pair at p. If (M,g,P) is $(r(p),s(p))$-infinitesimally P-homogeneous, then $(r(p),s(p))$ is a stabilizing pair at all $q \in M$ (so that we can omit the point p). In fact, any isometry $F : T_pM \to T_qM$ with $F^*(\nabla^i R_q) = \nabla^i R_p$ and $F^*(\nabla^j P_q) = \nabla^j P_p$ for $i = 0,\dots,r(p)+1$ and $j = 0,\dots,s(p)+1$, induces isomorphisms between $\mathfrak{h}(p,i,j)$ and $\mathfrak{h}(q,i,j)$ for $i \le r(p)$ and $j \le s(p)$. In addition, if (M,g,P) is $(k(p),l(p))$-infinitesimally P-homogeneous then the numbers $k(q)$ and $l(q)$ are independent of $q \in M$. Let $H(p,r,s)$ be the stabilizing group of the tensors $\nabla^i R_p$, and $\nabla^j P_p$, $0 \le i \le r+1$, $0 \le j \le s+1$, inside $O(T_pM)$. It is evident that $\mathfrak{h}(p,r,s)$ is the Lie algebra of $H(p,r,s)$.

Obviously, a locally homogeneous pseudo-Riemannian manifold with P invariant is in particular (r,s)-infinitesimally P-homogeneous for every pair (r,s). We shall see that the converse is also true.

Definition 3.2.2 *Let (r,s) be a stabilizing pair at $p \in M$. We say that (M,g,P) is (r,s)-strongly reductive at p if there is an $\mathrm{Ad}(H(p,r,s))$-invariant subspace $\mathfrak{n}(p,r,s) \subset \mathfrak{so}(T_pM)$ such that*

$$\mathfrak{so}(T_pM) = \mathfrak{h}(p,r,s) \oplus \mathfrak{n}(p,r,s).$$

Lemma 3.2.3 *Let (M,g,P) be an (r,s)-infinitesimally P-homogeneous manifold. If (M,g,P) is (r,s)-strongly reductive at $p \in M$, then it is (r,s)-strongly reductive at every point $q \in M$.*

Proof. Let $q \in M$ be another point distinct from p, recall that (r,s) is also a stabilizing pair at q. Let $F : T_pM \to T_qM$ be a linear isometry such that $F^*(\nabla^i R_q) = \nabla^i R_p$ and $F^*(\nabla^j P_q) = \nabla^j P_p$ for $i = 0,\dots,r+1$ and $j = 0,\dots,s+1$. F induces a linear isomorphism $\widetilde{F} : \mathfrak{so}(T_pM) \to \mathfrak{so}(T_qM)$ given by $A \mapsto F \circ A \circ F^{-1}$. By construction it is obvious that $\widetilde{F}(\mathfrak{h}(p,r,s)) = \mathfrak{h}(q,r,s)$. Let $\mathfrak{n}(p,r,s)$ be an $\mathrm{Ad}(H(p,r,s))$-invariant complement to $\mathfrak{h}(p,r,s)$, we define

$$\mathfrak{n}(q,r,s) = \widetilde{F}(\mathfrak{n}(p,r,s)) \subset \mathfrak{so}(T_qM).$$

This subspace is independent of the isometry F. Indeed, let $G : T_pM \to T_qM$ be another linear isometry with $G^*(\nabla^i R_q) = \nabla^i R_p$ and $G^*(\nabla^j P_q) = \nabla^j P_p$ for $i = 0,\dots,r+1$ and $j = 0,\dots,s+1$. The composition $G^{-1} \circ F$ is an element of $O(T_pM)$. Moreover, $G^{-1} \circ F$ stabilizes $R_p,\dots,\nabla^{r+1}R_p$ and $P_p,\dots,\nabla^{s+1}P_p$, so that it is an element of $H(p,r,s)$. Hence, for any $A \in \mathfrak{n}(p,r,s)$ we have

$$\widetilde{G}^{-1} \circ \widetilde{F}(A) = \mathrm{Ad}_{G^{-1}\circ F}(A) \in \mathfrak{n}(p,r,s),$$

showing that $\widetilde{F}(\mathfrak{n}(p,r,s))$ does not depend on the linear isometry F. We finally show that $\mathfrak{n}(q,r,s)$ is $\mathrm{Ad}(H(q,r,s))$-invariant. Let $B \in \mathfrak{n}(q,r,s)$, there is $A \in \mathfrak{n}(p,r,s)$ with $B = \widetilde{F}(A)$. Let $b \in H(q,r,s)$, we take $a = F^{-1} \circ b \circ F \in H(p,r,s)$. Then

$$\mathrm{Ad}_b(B) = b \circ B \circ b^{-1} = F \circ a \circ A \circ a^{-1} \circ F^{-1} = \widetilde{F}(\mathrm{Ad}_a(A)),$$

which belongs to $\mathfrak{n}(q,r,s)$ since $\mathrm{Ad}_a(A) \in \mathfrak{n}(p,r,s)$. ■

By virtue of the previous Lemma, we will say that an (r,s)-infinitesimally P-homogeneous manifold (M,g,P) is (r,s)-*strongly reductive* if it is (r,s)-strongly reductive at some point of M. The same applies for locally homogeneous spaces with P invariant. The term "strongly reductive" is motivated by Proposition 3.2.11 and Example 3.4.2, which show that strong reductivity implies reductivity but the converse is not true.

Remark 3.2.4 *In the case g is Riemannian, the Killing form of $\mathfrak{so}(T_pM)$ is definite, so that the strong reductivity condition is automatically satisfied choosing for $\mathfrak{n}(p,r,s)$ the orthogonal complement of $\mathfrak{h}(p,r,s)$ inside $\mathfrak{so}(T_pM)$ with respect to the Killing form. When the presence of an extra geometric structure is not taken into account, the integer $k(p)$ stabilizing the filtration*

$$\mathfrak{so}(T_pM) \supset \mathfrak{g}(p,0) \supset \ldots \supset \mathfrak{g}(p,r) \supset \ldots$$

is a pseudo-Riemannian invariant of (M,g) known as the Singer invariant. In this case, the choice of $\mathfrak{g}(p,k(p))^{\perp}$ as complement of $\mathfrak{g}(p,k(p))$ leads to the canonical AS-connection constructed by Kowalski in [40] in a similar way to the proof of Theorem 3.2.8 below.

Let $\pi : \mathcal{O}(M) \to M$ be the bundle of orthonormal frames with structure group $O(\nu, n-\nu)$, where ν is the index of the metric. Let $u_0 \in \mathcal{O}(M)$ with $\pi(u_0) = p$, and $P_0 = u_0^*(P_p)$. Let $\mathbf{P}$ be the space of tensors to which P_0 belongs. For any pair of integers $(r,s) \in (\mathbb{N} \cup \{0,-1\})^2$ consider the following $O(\nu, n-\nu)$-equivariant map:

$$\begin{array}{rccl} \Phi_{(r,s)} : & \mathcal{O}(M) & \to & \bigoplus_{i=0}^{k+1} \left(\bigotimes^{r+4} (\mathbb{R}^n)^*\right) \oplus \bigoplus_{j=0}^{s+1} \left(\bigotimes^{j} (\mathbb{R}^n)^* \otimes \mathbf{P}\right) \\ & u & \mapsto & u^*(R_{\pi(u)}, \ldots, \nabla^{r+1} R_{\pi(u)}, P_{\pi(u)}, \ldots, \nabla^{s+1} P_{\pi(u)}). \end{array}$$

Lemma 3.2.5 *Let (M,g,P) be (r,s)-infinitesimally P-homogeneous. Then $\Phi_{(r,s)}(\mathcal{O}(M))$ is a single $O(\nu, n-\nu)$-orbit.*

Proof. Let $u \in \mathcal{O}(M)$ and denote $\Phi = \Phi_{(r,s)}$. If $\pi(u_0) = \pi(u)$ then u_0 and u are in the same $O(\nu, n-\nu)$-orbit, and since Φ is $O(\nu, n-\nu)$-equivariant, we have that $\Phi(u_0)$ and $\Phi(u)$ are in the same $O(\nu, n-\nu)$-orbit. If $\pi(u_0) \neq \pi(u)$, let $q = \pi(u)$, then there is a linear isometry $F : T_pM \to T_qM$ such that $F^*(\nabla^i R_q) = \nabla^i R_p$ for $i = 0, \ldots, r+1$, and $F^*(\nabla^j P_q) = \nabla^j P_p$ for $j = 0, \ldots, s+1$. F induces a map $\widetilde{F} : \mathcal{O}(M) \to \mathcal{O}(M)$ such that $\Phi \circ \widetilde{F} = \Phi$. Since $\pi(u) = \pi(\widetilde{F}(u_0))$, we conclude that $\Phi(u_0)$ and $\Phi(u)$ are in the same $O(\nu, n-\nu)$-orbit. ∎

Lemma 3.2.6 *If (M,g,P) is an (r,s)-infinitesimally P-homogeneous manifold. Then there is a metric connection $\bar{\nabla}$ such that $\bar{\nabla}_X(\nabla^i R) = 0$ for $i = 0, \ldots, r+1$, and $\bar{\nabla}_X(\nabla^j P) = 0$ for $j = 0, \ldots, s+1$.*

Proof. Let $u_0 \in P$ with $\pi(u_0) = p$ and $\Phi = \Phi_{(r,s)}$. By Lemma 3.2.5 the orbit $\Phi(P)$ is the homogeneous space $O(\nu, n-\nu)/I_0$, where I_0 is the isotropy group at $\Phi(u_0)$. We thus have an equivariant map $\Phi : \mathcal{O}(M) \to O(\nu, n-\nu)/I_0$, so that $Q = \Phi^{-1}(\Phi(u_0))$ determines a reduction of $\mathcal{O}(M)$ with group I_0. Since Φ restricted to Q is constant, all the tensor fields $\nabla^i R$ and $\nabla^j P$, $i = 0, \ldots, r+1$, $j = 0, \ldots, s+1$, will be parallel with respect to any connection adapted to Q. ∎

Lemma 3.2.7 *If (M,g,P) is an (r,s)-infinitesimally P-homogeneous manifold, then $\mathfrak{h}(M,r,s) = \bigcup_{q\in M} \mathfrak{h}(q,r,s)$ is a vector subbundle of $\mathfrak{so}(M)$. If (M,g,P) is moreover (r,s)-strongly reductive, then $\mathfrak{n}(M,r,s) = \bigcup_{q\in M} \mathfrak{n}(q,r,s)$ is a vector subbundle of $\mathfrak{so}(M)$ and*

$$\mathfrak{so}(M) = \mathfrak{h}(M,r,s) \oplus \mathfrak{n}(M,r,s).$$

Proof. To prove that $\mathfrak{h}(M,r,s)$ is a vector subbundle of $\mathfrak{so}(M)$ we have to find a neighborhood U around every $q \in M$ with local sections $\{H_1, \ldots, H_t\}$ such that $\{H_1(y), \ldots, H_t(y)\}$ is a basis of $\mathfrak{h}(y,r,s)$ for every $y \in U$. Let $\bar{\nabla}$ be a linear connection as in Lemma 3.2.6, we take a normal neighborhood U around q with respect to $\bar{\nabla}$. Let $\{H_1(q), \ldots, H_t(q)\}$ be a basis of $\mathfrak{h}(q,r,s)$, we extend them by parallel transport with respect to $\bar{\nabla}$ along radial $\bar{\nabla}$-geodesics in order to define $\{H_1(y), \ldots, H_t(y)\}$.

Since $\bar{\nabla}_X(\nabla^i R) = 0$ for $i = 0, \ldots, r+1$, and $\bar{\nabla}_X(\nabla^j P) = 0$ for $j = 0, \ldots, s+1$, the parallel transport from q to y defines a linear isometry $F : T_qM \to T_yM$ with $F^*(\nabla^i R_y) = \nabla^i R_q$ for $i = 0, \ldots, r+1$, and $F^*(\nabla^j P_y) = \nabla^j P_q$ for $j = 0, \ldots, s+1$. This implies that $H_i(y) \in \mathfrak{h}(y, r, s)$. If (M, g, P) is (r, s)-strongly reductive, we consider the decomposition $\mathfrak{so}(T_qM) = \mathfrak{h}(q, r, s) \oplus \mathfrak{n}(q, r, s)$ and take a basis $\{\eta_1(q), \ldots, \eta_d(q)\}$ of $\mathfrak{n}(q, r, s)$. Extending $\{\eta_1(q), \ldots, \eta_d(q)\}$ by parallel transport along radial $\bar{\nabla}$-geodesics, we obtain local sections $\eta_1, \ldots, \eta_d$ of $\mathfrak{so}(M)$ defined on U. As seen in Lemma 3.2.3, the linear isometries F determined by the parallel transport takes $\mathfrak{n}(q, r, s)$ to $\mathfrak{n}(y, r, s)$ for $y \in U$, whence $\{\eta_1(y), \ldots, \eta_d(y)\}$ is a basis of $\mathfrak{n}(y, r, s)$ for every $y \in U$. ∎

Theorem 3.2.8 *Let (M, g, P) be an (r, s)-infinitesimally P-homogeneous manifold. If (M, g, P) is (r, s)-strongly reductive with a decomposition $\mathfrak{so}(T_pM) = \mathfrak{n}(p, r, s) \oplus \mathfrak{h}(p, r, s)$ with $\mathfrak{n}(M, r, s)$ $\mathrm{Ad}(H(p, r, s))$-invariant, then there is a unique ASK-connection $\widetilde{\nabla}$ such that $S = \widetilde{\nabla} - \nabla$ is a section of $T^*M \otimes \mathfrak{n}(M, r, s)$.*

Proof. Let $\mathfrak{h}(M)$ denote $\mathfrak{h}(M, r, s)$ and let $\mathfrak{n}(M)$ denote $\mathfrak{n}(M, r, s)$. Let $\bar{\nabla}$ be a linear connection as in Lemma 3.2.6. We consider the tensor field $B = \nabla - \bar{\nabla}$, which defines a section of $T^*M \otimes \mathfrak{so}(M)$ as $\bar{\nabla}$ is metric. By virtue of Lemma 3.2.7 we decompose

$$B = B^{\mathfrak{h}} + B^{\mathfrak{n}},$$

with $B^{\mathfrak{h}}$ and $B^{\mathfrak{n}}$ sections of $T^*M \otimes \mathfrak{h}(M)$ and $T^*M \otimes \mathfrak{n}(M)$ respectively. We define $S = B^{\mathfrak{n}}$, and take $\widetilde{\nabla} = \nabla - S$. Since S is a section of $T^*M \otimes \mathfrak{so}(M)$ we have that $\widetilde{\nabla}$ is metric, so that $\widetilde{\nabla} g = 0$. Moreover

$$\widetilde{\nabla}_X(\nabla^i R) = \bar{\nabla}_X(\nabla^i R) + B^{\mathfrak{h}}_X \cdot (\nabla^i R) = 0, \qquad i = 0, \ldots, r+1,$$

$$\widetilde{\nabla}_X(\nabla^j P) = \bar{\nabla}_X(\nabla^j P) + B^{\mathfrak{h}}_X \cdot (\nabla^j P) = 0, \qquad j = 0, \ldots, s+1,$$

since (r, s) is a stabilizing pair. Let $q \in M$ and consider a normal neighborhood of q with respect to $\widetilde{\nabla}$. Since

$$0 = \widetilde{\nabla}_X(\nabla^i R) = i_X(\nabla^{i+1} R) - S_X \cdot (\nabla^i R),$$

$$0 = \widetilde{\nabla}_X(\nabla^j P) = i_X(\nabla^{j+1} P) - S_X \cdot (\nabla^j P),$$

differentiating these formulas along a radial $\widetilde{\nabla}$-geodesic $\gamma(t)$ we find

$$0 = 0 - \frac{d}{dt}\left(S_{\dot{\gamma}(t)} \cdot (\nabla^i R)_{\gamma(t)}\right) = -\left(\widetilde{\nabla}_{\dot{\gamma}(t)} S\right) \cdot (\nabla^i R)_{\gamma(t)},$$

$$0 = 0 - \frac{d}{dt}\left(S_{\dot{\gamma}(t)} \cdot (\nabla^j P)_{\gamma(t)}\right) = -\left(\widetilde{\nabla}_{\dot{\gamma}(t)} S\right) \cdot (\nabla^j P)_{\gamma(t)},$$

for $i = 0, \ldots, r$ and $j = 0, \ldots, s$. This means that $\widetilde{\nabla}_{\dot{\gamma}(t)} S \in \mathfrak{h}(\gamma(t), r, s)$. In addition, as a consequence of the $\mathrm{ad}(\mathfrak{h}(M))$-invariance of $\mathfrak{n}(M)$, the covariant derivative of a section of $\mathfrak{n}(M)$ is again a section of $\mathfrak{n}(M)$, so that $\widetilde{\nabla}_{\dot{\gamma}(t)} S \in \mathfrak{n}(\gamma(t), r, s)$. We conclude that $\widetilde{\nabla} S = 0$.

We finally prove uniqueness. Let $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ be as in the hypothesis, then $S - S'$ is a section of $T^*M \otimes \mathfrak{n}(M)$. In addition $\widetilde{\nabla}(\nabla^i R) = \widetilde{\nabla}'(\nabla^i R) = 0$ and $\widetilde{\nabla}(\nabla^j P) = \widetilde{\nabla}'(\nabla^j P) = 0$ for all i, j. These are easily obtained from the fact that the torsion and the curvature of $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}'$) are parallel with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}'$), and from $\widetilde{\nabla} P = \widetilde{\nabla}' P = 0$. This implies that $S - S'$ is a section of $T^*M \otimes \mathfrak{h}(M)$, and then $S = S'$ and $\widetilde{\nabla} = \widetilde{\nabla}'$. ∎

Corollary 3.2.9 *If $\mathfrak{n}(p, r, s) \subset \mathfrak{n}(p, r', s')$ for stabilizing pairs (r, s) and (r', s'). Then the connections $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ constructed from them coincide.*

Proof. This is evident since $S = \widetilde{\nabla} - \nabla$ is a section of both $\mathfrak{n}(M, r, s)$ and $\mathfrak{n}(M, r', s')$. ■

As we have seen, a strongly reductive locally homogeneous pseudo-Riemannian manifold (M, g, P) with P invariant admits an ASK-connection, so by Theorem 3.1.10 there is a Lie pseudo-group $\mathcal{G}$ (which is not necessarily the full isometry pseudo-group) acting transitively by isometries and preserving P such that $(M, g, P, \mathcal{G})$ is reductive. Moreover, we shall show that strongly reductive locally homogeneous spaces with an invariant geometric structure P are reductive for the action of the full pseudo-group of isometries preserving P. In order to prove that we will make use of some results contained in Section 3.3 and the following Lemma.

Lemma 3.2.10 *Let $\widetilde{\nabla}$ be an ASK-connection with curvature K and torsion T. Let $p \in M$, and let $A \in \mathfrak{so}(T_pM)$ be such that $A \cdot K_p = 0$, $A \cdot T_p = 0$ and $A \cdot P_p = 0$. Then $A \cdot \nabla^i R_p = 0$ and $A \cdot \nabla^j P_p = 0$ for all $i, j \geq 0$.*

Proof. The curvature and torsion of $\widetilde{\nabla}$ are related to R and S by

$$T_X Y = S_Y X - S_X Y, \qquad K_{XY} = R_{XY} + [S_X, S_Y] + S_{T_X Y}.$$

Making use of these formulas in conjunction with $\widetilde{\nabla} R = 0$ and $\widetilde{\nabla} S = 0$, an inductive argument gives that $\widetilde{\nabla}(\nabla^i R) = 0$ for all $i \geq 0$. A similar computation gives $\widetilde{\nabla}(\nabla^j P) = 0$ for all $j \geq 0$. This means that

$$i_X \nabla^{i+1} R = S_X \cdot \nabla^i R, \qquad i_X \nabla^{j+1} P = S_X \cdot \nabla^j P,$$

for all $i, j \geq 0$. Let now $A \in \mathfrak{so}(T_pM)$ be such that $A \cdot K_p = 0$, $A \cdot T_p = 0$ and $A \cdot P_p = 0$. By Corollary 3.3.5 $A \cdot S_p = 0$, hence $A \cdot R_p = 0$. A simple computation thus leads to

$$(A \cdot \nabla^{i+1} R_p)_X = (A \cdot S_p)_X \cdot \nabla^i R_p + (S_p)_X \cdot (A \cdot \nabla^i R_p), \qquad i \geq 0,$$

$$(A \cdot \nabla^{j+1} P_p)_X = (A \cdot S_p)_X \cdot \nabla^j P_p + (S_p)_X \cdot (A \cdot \nabla^j P_p), \qquad j \geq 0.$$

Therefore, by induction on i and j we obtain that $A \cdot \nabla^i R_p = 0$ and $A \cdot \nabla^j P_p = 0$ for all $i, j \geq 0$. ■

Proposition 3.2.11 *If (M, g, P) is (r, s)-strongly reductive, then $(M, g, \mathcal{J})$ is reductive with $\mathcal{J}$ the full Lie pseudo-group of local isometries preserving P.*

Proof. Let $\mathfrak{so}(T_pM) = \mathfrak{n}(p, r, s) \oplus \mathfrak{h}(p, r, s)$, and let $\widetilde{\nabla}$ be the associated ASK-connection. Let K and T be the curvature and the torsion tensor fields of $\widetilde{\nabla}$ respectively. The triple (K, T, P) defines an infinitesimal model (see Proposition 3.3.6), and we can consider the associated Nomizu construction, that is, we define the Lie algebra $\mathfrak{g}_0 = T_pM \oplus \mathfrak{h}_0$ with the usual brackets, where

$$\mathfrak{h}_0 = \{A \in \mathfrak{so}(T_pM) / \, A \cdot K_p = 0, \, A \cdot T_p = 0, \, A \cdot P_p = 0\}.$$

By Proposition 3.3.7 the Lie algebra $\mathfrak{h}_0$ is equal to $\mathfrak{h}(p, r, s)$. On the other hand, $\mathfrak{h}_0 \subset \mathfrak{gkill}^0$ by Lemma 3.2.10, and $\mathfrak{gkill}^0 \subset \mathfrak{h}$ by definition, whence $\mathfrak{gkill}^0 \subset \mathfrak{h} = \mathfrak{h}_0$. We thus define the following Lie algebra isomorphism

$$\begin{array}{rccl} \Phi: & \mathfrak{g}_0 & \to & \mathfrak{gkill} \\ & X + A & \mapsto & (X, (S_0)_X + A). \end{array}$$

The image of T_pM defines a complement $\mathfrak{m}$ of $\mathfrak{gkill}^0$. Making use of Lemma 3.3.4 we have that $\mathrm{Ad}_B(S_X) = S_{BX}$ for all B in $H(p, r, s)$ and all $X \in T_pM$. Since the linear isotropy group H_p is contained in $H(p, r, s)$ we have that $\mathfrak{m}$ is $\mathrm{Ad}(H_p)$-invariant. ■

3.3 Reconstruction of strongly reductive locally homogeneous spaces

We first show a uniqueness result satisfied by strongly reductive locally homogeneous manifolds.

Proposition 3.3.1 *Let (M, g, P) and (M', g', P') be pseudo-Riemannian manifolds endowed with tensor fields P and P'. Suppose (M', g', P') is locally homogeneous with P' invariant. Suppose furthermore that (M', g', P') is (r, s)-strongly reductive for some stabilizing pair (r, s). If for each point $p \in M$ there is a linear isometry $F : T_pM \to T_oM'$ (where $o \in M'$ can be fixed) such that $F^*(\nabla'^i R'_o) = \nabla^i R_p$ for $i = 0, \ldots, r + 1$, and $F^*(\nabla'^j P_o) = \nabla^j P_p$ for $j = 0, \ldots, s + 1$. Then (M, g, P) is locally homogeneous with P invariant and locally isometric to (M', g', P') preserving P and P'.*

Proof. Note first of all that (M, g, P) is (r, s)-infinitesimally P-homogeneous and (r, s)-strongly reductive, so that (M, g, P) is locally homogeneous with P invariant. Let $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ be connections on M and M' respectively as in Theorem 3.2.8. Let $S = \nabla - \widetilde{\nabla}$ and $S' = \nabla' - \widetilde{\nabla}'$, and let $F : T_pM \to T_oM'$ as in the hypothesis. It is obvious that $F^*(S'_o) - S_p \in T^*_pM \otimes \mathfrak{n}(p, r, s)$. In addition

$$(F^*(S'_o)_X - S_{pX}) \cdot (\nabla^i R_p) = i_X \nabla^{i+1} R_p - i_X \nabla^{i+1} R_p = 0, \qquad i = 0, \ldots, r,$$

$$(F^*(S'_o)_X - S_{pX}) \cdot (\nabla^j P_p) = i_X \nabla^{j+1} P_p - i_X \nabla^{j+1} P_p = 0, \qquad j = 0, \ldots, s,$$

so that $F^*(S'_o)_X - S_{pX} \in \mathfrak{h}(p, r, s)$. We conclude that $F^*(S'_o) = S_p$. Since the torsion of $\widetilde{\nabla}$ is $S_Y X - S_X Y$ and a similar formula holds for the torsion of $\widetilde{\nabla}'$, as a simple inspection shows, F preserves the curvature and the torsion of $\widetilde{\nabla}$ and $\widetilde{\nabla}'$, which are parallel with respect to $\widetilde{\nabla}$. Therefore, there are neighborhoods U and V around p and o respectively, and an affine map $f : U \to V$ with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ (see [38, Ch. 7]). Since $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ are metric and $\widetilde{\nabla} P = \widetilde{\nabla}' P' = 0$ we have that f is an isometry preserving P and P'. ■

Theorem 3.2.8 and Proposition 3.3.1 suggest the possibility of reconstructing a strongly reductive locally homogeneous manifold (M, g, P) with P invariant from the knowledge of the curvature tensor field, the tensor field P, and their covariant derivatives up to finite order. In order to prove this result we must first examine the algebraic properties of the curvature tensor field, P and its covariant derivatives.

Let (M, g, P) be a locally homogeneous manifold with P invariant. We fix a point $p \in M$ and set $V = T_pM$. Consider the tensors $R^i = \nabla^i R_p$ and $P^j = \nabla^j P_p$ for $i, j \geq 0$. One has

$$R^0_{XYZW} = -R^0_{YXZW} = R^0_{ZWXY}, \tag{3.3}$$

$$\underset{XYZ}{\mathfrak{S}} R^0_{XYZW} = 0, \tag{3.4}$$

$$R^1_{XYZVW} = -R^1_{XZYVW} = R^1_{XVWYZ}, \tag{3.5}$$

$$\underset{YZV}{\mathfrak{S}} R^1_{XYZVW} = 0, \tag{3.6}$$

$$\underset{XYZ}{\mathfrak{S}} R^1_{XYZVW} = 0, \tag{3.7}$$

$$R^{i+2}_{YX} - R^{i+2}_{XY} = R^0_{XY} \cdot R^i, \tag{3.8}$$

$$P^{j+2}_{YX} - P^{j+2}_{XY} = R^0_{XY} \cdot P^j, \tag{3.9}$$

for $i, j \geq 0$, where R^0_{XY} is acting as a derivation on the tensor algebra. In addition, let $\widetilde{\nabla}$ be an ASK-connection and $S = \nabla - \widetilde{\nabla}$. We have that

$$i_X R^{i+1} = S_X \cdot R^i, \qquad i_X P^{j+1} = S_X \cdot P^j,$$

for $0 \leq i \leq r+1$, $0 \leq j \leq s+1$, where (r, s) is a stabilizing pair at p. We thus consider the following linear maps

$$\begin{array}{rrcl} \mu_{i,j}: & \mathfrak{so}(V) & \to & W_{i,j} \\ & A & \mapsto & (A \cdot R^0, \ldots, A \cdot R^i, A \cdot P^0, \ldots, A \cdot P^j), \end{array}$$

and

$$\begin{array}{rrcl} \nu: & V & \to & W_{r+1,s+1} \\ & X & \mapsto & (i_X R^1, \ldots, i_X R^{r+2}, i_X P^1, \ldots, i_X P^{s+2}), \end{array}$$

with

$$W_{i,j} = \left[\bigoplus_{\alpha=0}^{i} (\otimes^{\alpha+4} V^*) \right] \otimes \left[\bigoplus_{\beta=0}^{j} ((\otimes^{\beta} V^*) \otimes \mathbf{P}) \right],$$

where $\mathbf{P}$ is the space of tensors to which P^0 belongs. The previous discussion for a stabilizing pair (r, s) thus gives

$$\nu(V) \subset \mu_{r+1,s+1}(\mathfrak{so}(V)), \tag{3.10}$$

and

$$ker(\mu_{r,s}) = ker(\mu_{r+1,s}) = ker(\mu_{r,s+1}) = ker(\mu_{r+1,s+1}). \tag{3.11}$$

Finally, let $H(r, s)$ be the stabilizer of $R^0, \ldots, R^{r+1}$ and $P^0, \ldots, P^{s+1}$ inside $O(V)$. In view of Theorem 3.2.8, to assure the existence of an ASK-connection we need that

$$\mathfrak{so}(V) = ker(\mu_{r,s}) \oplus \mathfrak{n} \tag{3.12}$$

for an $\mathrm{Ad}(H(r, s))$-invariant subspace $\mathfrak{n}$. We shall prove the following result.

Theorem 3.3.2 *Let V be a vector space endowed with an inner product $\langle\, , \rangle$. Let $R^0, \ldots, R^{r+2}$, $P^0, \ldots, P^{s+2}$ be tensors on V satisfying* (3.3),...,(3.9) *for $0 \leq i \leq r$ and $0 \leq j \leq s$, and such that* (3.10), (3.11), *and* (3.12) *hold. Then*

1. *There is an (r, s)-strongly reductive locally homogeneous pseudo-Riemannian manifold (M, g, P) with P invariant, whose curvature tensor field, P, and their covariant derivatives coincide with $R^0, \ldots, R^{r+2}$, $P^0, \ldots, P^{s+2}$ at a point $p \in M$. Moreover, (M, g, P) is unique up to local isometry preserving P.*

2. *If the infinitesimal data $R^0, \ldots, R^{r+2}$, $P^0, \ldots, P^{s+2}$ is regular (see Definitions 2.3.3 and 3.3.8), then there is an (r, s)-strongly-reductive globally homogeneous pseudo-Riemannian space $(G_0/H_0, g, P)$, whose curvature tensor field, P, and their covariant derivatives coincide with $R^0, \ldots, R^{r+2}$, $P^0, \ldots, P^{s+2}$ at a point $p \in M$. Moreover, $(G_0/H_0, g, P)$ is unique up to local isometry preserving P.*

Corollary 3.3.3 *An (r, s)-strongly-reductive locally homogeneous Riemannian manifold (M, g, P) with P invariant can be reconstructed (up to local isometry) from the data $R_p, \ldots, \nabla^{r+2} R_p$, $P_p, \ldots, \nabla^{s+2} P_p$, where (r, s) is a stabilizing pair at $p \in M$.*

Before proving Theorem 3.3.2 we need to recall the definition of *infinitesimal model* (see Definition 2.3.3) and show that an infinitesimal model can be associated to every suitable infinitesimal data $R^0, \ldots, R^{s+2}$, $P^0, \ldots, P^{r+2}$ satisfying the hypotheses of Theorem 3.3.2. We define $\mathfrak{h} = ker(\mu_{r+1,s+1})$, and consider an $Ad(H(r, s))$-invariant complement $\mathfrak{n}$ of $\mathfrak{h}$ inside $\mathfrak{so}(V)$. From (3.10) we have that for every $X \in V$ there is an endomorphism $A(X) \in \mathfrak{so}(V)$ such that

$$\begin{aligned} i_X R^{i+1} &= A(X) \cdot R^i, \qquad 0 \le i \le r+1, \\ i_X P^{j+1} &= A(X) \cdot P^j, \qquad 0 \le j \le s+1. \end{aligned}$$

Since $\mathfrak{so}(V) = \mathfrak{h} \oplus \mathfrak{n}$, we decompose $A(X) = A_1(X) + A_2(X)$, where $A_1(X) \in \mathfrak{h}$ and $A_2(X) \in \mathfrak{n}$. Note that $A(X)$ is uniquely determined up to an $\mathfrak{h}$-component, so that we can take the uniquely defined map

$$\begin{array}{rccl} S: & V & \to & \mathfrak{n} \\ & X & \mapsto & S_X = A_2(X). \end{array}$$

By the definition of $\mathfrak{h}$ it is evident that

$$i_X R^{i+1} = S_X \cdot R^i, \qquad 0 \le i \le r+1, \tag{3.13}$$
$$i_X P^{j+1} = S_X \cdot P^j, \qquad 0 \le j \le s+1. \tag{3.14}$$

Moreover, by the same arguments used in [49] one sees that S is a linear map.

Lemma 3.3.4 *Let $B \in H(r,s)$, then $\mathrm{Ad}_B(S_X) = S_{BX}$ for every $X \in V$.*

Proof. By the definition of $H(r,s)$, (3.13) and (3.14), we have for $0 \le i \le r$ and $0 \le j \le s$

$$\begin{aligned} R^{i+1}_{XZ_1\dots Z_{i+4}} &= (B \cdot R^{i+1})_{XZ_1\dots Z_{i+4}} \\ &= R^{i+1}_{B^{-1}XB^{-1}Z_1\dots B^{-1}Z_{i+4}} \\ &= \left(S_{B^{-1}X} \cdot R^i\right)_{B^{-1}Z_1\dots B^{-1}Z_{i+4}} \\ &= -\sum_\alpha R^i_{B^{-1}Z_1\dots S_{B^{-1}X}B^{-1}Z_\alpha\dots B^{-1}Z_{i+4}} \\ &= -\sum_\alpha R^i_{B^{-1}Z_1\dots B^{-1}BS_{B^{-1}X}B^{-1}Z_\alpha\dots B^{-1}Z_{i+4}} \\ &= -\sum_\alpha (B \cdot R^i)_{Z_1\dots \mathrm{Ad}_B(S_{B^{-1}X})Z_\alpha\dots Z_{i+4}} \\ &= -\sum_\alpha R^i_{Z_1\dots \mathrm{Ad}_B(S_{B^{-1}X})Z_\alpha\dots Z_{i+4}} \\ &= \left(\mathrm{Ad}_B(S_{B^{-1}X}) \cdot R^i\right)_{Z_1\dots Z_{i+4}}. \end{aligned}$$

On the other hand $i_X R^{i+1} = S_X \cdot R^i$, so that $\mathrm{Ad}_B(S_{B^{-1}X}) \cdot R^i - S_X$ belongs to $\mathfrak{h}$. Since S_X belongs to $\mathfrak{n}$ and $\mathfrak{n}$ is $\mathrm{Ad}(H(r,s))$-invariant, we also have that $\mathrm{Ad}_B(S_{B^{-1}X}) \cdot R^i - S_X$ belongs to $\mathfrak{n}$. This implies that $\mathrm{Ad}_B(S_{B^{-1}X}) \cdot R^i - S_X = 0$. ∎

Corollary 3.3.5 *Let $A \in \mathfrak{h}$, then $A \cdot S = 0$.*

We take

$$\begin{aligned} T_X Y &= S_Y X - S_X Y, \\ K_{XY} &= R^0_{XY} + [S_X, S_Y] + S_{T_X Y}, \\ P &= P^0. \end{aligned}$$

Proposition 3.3.6 *The triple (K,T,P) is an infinitesimal model.*

Proof. We have to prove that (T, K, P) satisfies (2.3),...,(2.9). For (2.3), (2.4), (2.5), (2.8) and (2.9) one uses exactly the same arguments used in [49]. For the remaining, we observe that

$$\begin{aligned} R^{i+2}_{XY} - R^{i+2}_{YX} &= ([S_X, S_Y] + S_{T_X Y}) \cdot R^i, \qquad 0 \le i \le r, \\ P^{j+2}_{XY} - P^{j+2}_{YX} &= ([S_X, S_Y] + S_{T_X Y}) \cdot P^j, \qquad 0 \le j \le s. \end{aligned}$$

In fact, by (3.13)

$$\begin{aligned} R^{i+2}_{XYZ_1 \dots Z_{i+4}} &= (i_X R^{i+2})_{YZ_1 \dots Z_i} = (S_X \cdot R^{i+1})_{XYZ_1 \dots Z_{i+4}} \\ &= -R^{i+1}_{S_X Y Z_1 \dots Z_{i+4}} - \sum_{\alpha=1}^{i+4} R^{i+1}_{YZ_1 \dots S_X Z_\alpha \dots Z_{i+4}} \\ &= -\left(i_{S_X Y} R^{i+1}\right)_{Z_1 \dots Z_{i+4}} - \sum_{\alpha=1}^{i+4} \left(i_Y R^{i+1}\right)_{Z_1 \dots S_X Z_\alpha \dots Z_{i+4}} \\ &= -\left(S_{S_X Y} \cdot R^i\right)_{Z_1 \dots Z_{i+4}} - \sum_{\alpha=1}^{i+4} \left(S_Y \cdot R^i\right)_{Z_1 \dots S_X Z_\alpha \dots Z_{i+4}} \\ &= \sum_{\alpha=1}^{i+4} R^i_{Z_1 \dots S_{S_X Y} Z_\alpha \dots Z_{i+4}} + \sum_{\alpha,\beta=1}^{i+4} R^i_{Z_1 \dots S_X Z_\alpha \dots S_Y Z_\beta \dots Z_{i+4}}, \end{aligned}$$

and by (3.14) a similar argument holds for P^{j+2}_{XY}. Skew-symmetrizing in X, Y we obtain the desired formulas. Therefore, by (3.8) and (3.9) and the definition of K we obtain that $K_{XY} \cdot R^i = 0$ and $K_{XY} \cdot P^j = 0$, for $0 \le i \le r$ and $0 \le j \le s$, so in particular $K_{XY} \cdot P^0 = 0$ and $K_{XY} \cdot R^0 = 0$. Making use of (3.11) this implies that $K_{XY} \in \mathfrak{h}$, whence by Corollary 3.3.5 $K_{XY} \cdot S = 0$, giving that $K_{XY} \cdot T = 0$. Finally as a straightforward computation shows, for $A \in \mathfrak{h}$

$$(A \cdot K)_{XY} = (A \cdot R^0)_{XY} + [(A \cdot S)_X, S_Y] - [(A \cdot S)_Y, S_X] + S_{(A \cdot T)_X Y}, \tag{3.15}$$

so that $K_{XY} \cdot K = 0$. ∎

Proposition 3.3.7

$$\mathfrak{h} = \mathfrak{h}_0 = \{A \in \mathfrak{so}(V) / \, A \cdot K = 0, \, A \cdot T = 0, \, A \cdot P = 0\}.$$

Proof. Let $A \in \mathfrak{h}$, by Corollary 3.3.5 we have $A \cdot S = 0$, which implies $A \cdot T = 0$. In addition, by (3.15) we have $A \cdot K = 0$. Since $P = P^0$ by definition we deduce that $A \in \mathfrak{h}_0$, hence $\mathfrak{h} \subset \mathfrak{h}_0$. Conversely, let $A \in \mathfrak{h}_0$. We have that $A \cdot S = 0$ since S is recovered from T making use of

$$2\langle S_X Y, Z \rangle = -\langle T_X Y, Z \rangle + \langle T_Y Z, X \rangle - \langle T_Z X, Y \rangle.$$

On the other hand, by (3.15) we obtain $A \cdot R^0 = 0$, and since $P = P^0$ we also have $A \cdot P^0 = 0$. Now, a simple calculation (see Lemma 3.3.4) shows that

$$\begin{aligned} (A \cdot R^{i+1})_X &= [A, S_X] \cdot R^i - S_{AX} \cdot R^i + S_X \cdot (A \cdot R^i) \\ &= (A \cdot S)_X \cdot R^i + S_X \cdot (A \cdot R^i), \qquad 0 \le i \le r+1, \\ (A \cdot P^{j+1})_X &= [A, S_X] \cdot P^j - S_{AX} \cdot P^j + S_X \cdot (A \cdot P^j) \\ &= (A \cdot S)_X \cdot P^j + S_X \cdot (A \cdot P^j), \qquad 0 \le j \le s+1. \end{aligned}$$

Using these formulas, by an inductive argument on the indices i and j we obtain that $A \cdot R^i = 0$ and $A \cdot P^j = 0$ for $0 \le i \le r+1$ and $0 \le j \le s+1$. Hence $A \in \mathfrak{h}$, proving $\mathfrak{h}_0 \subset \mathfrak{h}$. ∎

Definition 3.3.8 *The infinitesimal data $R^0, \ldots, R^{r+2}$, $P^0, \ldots, P^{s+2}$ will be called regular if the associated infinitesimal model (T, K, P) is regular.*

Remark 3.3.9 *$R^0, \ldots, R^{r+2}$, $P^0, \ldots, P^{s+2}$ is recovered from the infinitesimal model (T, K, P) in the following way. As we have seen, S is obtained from T by*

$$2\langle S_X Y, Z\rangle = -\langle T_X Y, Z\rangle + \langle T_Y Z, X\rangle - \langle T_Z X, Y\rangle.$$

With T and S one recovers R^0 using the definition of K. Finally, knowing R^0 and $P^0 = P$, and using (3.13) *and* (3.14)*, one can subsequently obtain R^i and P^j.*

We are now in position to prove Theorem 3.3.2.

Proof of Theorem 3.3.2. Suppose that the infinitesimal model (T, K, P) associated to the infinitesimal data $R^0, \ldots, R^{r+2}$, $P^0, \ldots, P^{s+2}$ is regular. We consider the Nomizu construction $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus V$, and the Lie groups G_0 and H_0, where G_0 is the simply-connected Lie group with Lie algebra $\mathfrak{g}_0$ and H_0 is its connected Lie subgroup with Lie algebra $\mathfrak{h}_0$. Since H_0 is closed in G_0 we consider the homogeneous space G_0/H_0. It is a reductive homogeneous space with reductive decomposition $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus V$, and identifying $V = T_o G_0/H_0$, where $o = H_0$ is the origin of G_0/H_0, we extend $\langle\, ,\, \rangle$ and P to a G_0-invariant Riemannian metric g and a G_0-invariant tensor field $\bar{P}$ on G_0/H_0 respectively. We consider the canonical connection associated to that reductive decomposition, which is an ASK-connection whose curvature and torsion at the origin o coincide with K and T. As a straightforward computation using the properties of the canonical connection shows, $R^0, \ldots, R^{r+2}$, $P^0, \ldots, P^{s+2}$ coincide with the covariant derivatives of the curvature of g and $\bar{P}$ at the origin o. By the identification of $T_o G_0/H_0$ with V, we have that G_0/H_0 is (r, s)-strongly reductive. This proves the second part of the Theorem.

Concerning the first part of the Theorem, we adapt the arguments used in [61]. Let (T, K, P) be the infinitesimal model associated to the infinitesimal data $R^0, \ldots, R^{r+2}$, $P^0, \ldots, P^{s+2}$, which now need not be regular. We consider the corresponding Nomizu construction $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus V$. Let G_0 be the simply-connected Lie group with Lie algebra $\mathfrak{g}_0$, we choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of V, and denote by $\{e^1, \ldots, e^n\}$ its dual basis. Let $\{A_1, \ldots, A_d\}$ be a basis of $\mathfrak{h}_0$, and $\{A^1, \ldots, A^d\}$ its dual basis. With respect to these basis we write

$$\begin{aligned}
T &= T^{\gamma}_{\alpha\beta} e^\alpha \otimes e^\beta \otimes e_\gamma, \\
K &= K^{\delta}_{\alpha\beta\gamma} e^\alpha \otimes e^\beta \otimes e^\gamma \otimes e_\delta, \\
P &= P^{\beta_1 \ldots \beta_v}_{\alpha_1 \ldots \alpha_u} e^{\alpha_1} \otimes \ldots \otimes e^{\alpha_u} \otimes e_{\beta_1} \otimes \ldots \otimes e_{\beta_v},
\end{aligned}$$

and define

$$\omega^\alpha_\beta = e^\alpha(A_\gamma(e_\beta)) \otimes A^\gamma,$$

where Einstein's summation convention is used. Note that $\omega^\alpha_\beta \in \mathfrak{g}_0^*$, so that

$$\omega = \omega^\alpha_\beta A_\alpha \otimes A^\beta$$

defines a left invariant 2-form on G_0 with values in $\mathfrak{h}_0 \subset \mathfrak{so}(V)$. Making use of the brackets defined in $\mathfrak{g}_0$ we easily obtain

$$de^\alpha = \frac{1}{2} T^\alpha_{\beta\gamma} - \omega^\alpha_\beta \wedge e^\beta, \tag{3.16}$$

$$d\omega^\alpha_\beta = -\frac{1}{2} K^\alpha_{\gamma\delta\beta} e^\gamma e^\delta - \omega^\alpha_\gamma \wedge \omega^\gamma_\beta. \tag{3.17}$$

We now consider a coordinate system $\phi = (x^1, \ldots, x^n, y^1, \ldots, y^d)$ around the identity element $e \in G_0$ such that $dx^\alpha_{|e} = e^\alpha_{|e}$, and take

$$\begin{array}{rrcl} f: & \widetilde{\mathcal{U}} & \to & \mathcal{U} \\ & (a_1, \ldots, a_n) & \mapsto & \phi^{-1}(a_1, \ldots, a_n, 0, \ldots, 0), \end{array}$$

where $\mathcal{U}$ is the coordinate neighborhood and $\widetilde{\mathcal{U}}$ is an open subset of $\mathbb{R}^n$ where f can be defined. It is evident that the map f defines an immersion from an open set $W \subset \mathbb{R}^n$ containing the origin of $\mathbb{R}^n$ into G_0. Let $\widetilde{E}^\alpha = f^*(e^\alpha)$, since these 1-forms are linearly independent at the origin of $\mathbb{R}^n$, there is an open set $M \subset W$ around the origin where they are linearly independent. Let $\{\widetilde{E}_1, \ldots, \widetilde{E}_n\}$ be the dual frame field, we define on M the pseudo-Riemannian metric

$$g = \sum_{\alpha=1}^{n} \widetilde{E}^\alpha \otimes \widetilde{E}^\alpha,$$

and the tensor fields

$$\begin{aligned} \widetilde{T} &= T^\gamma_{\alpha\beta} \widetilde{E}^\alpha \otimes \widetilde{E}^\beta \otimes \widetilde{E}_\gamma, \\ \widetilde{K} &= K^\delta_{\alpha\beta\gamma} \widetilde{E}^\alpha \otimes \widetilde{E}^\beta \otimes \widetilde{E}^\gamma \otimes \widetilde{E}_\delta, \\ \widetilde{P} &= P^{\beta_1 \ldots \beta_v}_{\alpha_1 \ldots \alpha_u} \widetilde{E}^{\alpha_1} \otimes \ldots \otimes \widetilde{E}^{\alpha_u} \otimes \widetilde{E}_{\beta_1} \otimes \ldots \otimes \widetilde{E}_{\beta_v}. \end{aligned}$$

In addition we consider $\widetilde{\omega} = f^*\omega$ which is a 1-form on M with values in $\mathfrak{h}_0$. Note that $\{\widetilde{E}_1, \ldots, \widetilde{E}_n\}$ is an orthonormal frame field defined on the whole M, so that it is a trivializing section of the bundle of orthonormal frames of M. Hence, this section $\widetilde{\omega}$ is the 1-form of a metric connection $\widetilde{\nabla}$ on $\mathcal{O}(M)$. By (3.16) and (3.17), which are nothing but the structure equations for the torsion and curvature of ω, we have that $\widetilde{T}$ and $\widetilde{K}$ are the torsion and curvature of the connection $\widetilde{\nabla}$ respectively. Since $\widetilde{\omega}$ takes values in $\mathfrak{h}_0$, we have that $\widetilde{T}$, $\widetilde{K}$ and $\widetilde{P}$ are parallel with respect to $\widetilde{\nabla}$, that is, $\widetilde{\nabla}$ is an ASK-connection. Therefore, $(M, g, \widetilde{P})$ is locally homogeneous with $\widetilde{P}$ invariant. Finally, making use of Remark 3.3.9, it is easy to see that the covariant derivatives of $\widetilde{P}$ and the curvature of g at the origin coincide with $R^0, \ldots, R^{r+2}$, $P^0, \ldots, P^{s+2}$ under the identification $T_oM \simeq V$. In addition, by this identification M is (r,s)-strongly reductive.

In both, the first and the second part of the theorem, uniqueness (up to local isometry) follows from Proposition 3.3.1. ■

Note that the strong reductivity condition (3.12) is essential in the proof of Theorem 3.3.2, since otherwise we are not able to construct the infinitesimal model (T, K, P) from the infinitesimal data $R^0, \ldots, R^{r+2}$, $P^0, \ldots, P^{s+2}$. This means that in general a locally homogeneous pseudo-Riemannian manifold whose metric is not definite might not be recovered from infinitesimal data. If the manifold admits an ASK-connection $\widetilde{\nabla}$, this problem can be solved if we add to $R^0, \ldots, R^{r+2}$, $P^0, \ldots, P^{s+2}$ the knowledge of either S_p, where $S = \widetilde{\nabla} - \nabla$, the torsion of $\widetilde{\nabla}$ at p, or the curvature of $\widetilde{\nabla}$ at p (these three last items provide equivalent information in view of Remark 3.3.9). In that case, an analogous result to Proposition 3.3.1 can be proved by a straightforward adaptation.

3.4 Examples and the reductivity condition

As we know, a globally homogeneous space can be represented as different coset spaces G/H. In the same way, we can consider the action of different Lie pseudo-groups of isometries on the same locally homogeneous pseudo-Riemannian manifold (M, g). Since the notion of reductivity is tied to the action of a Lie pseudo-group in particular, the following question naturally arises: let $\mathcal{G}$ and $\mathcal{G}'$ be Lie pseudo-groups of isometries acting transitively on (M,g), is it possible that $(M,g,\mathcal{G})$ is reductive but $(M,g,\mathcal{G}')$ is non-reductive? We now present some examples which give an affirmative answer to this question, and explores the possible scenarios when $\mathcal{G}$ is a subgroup of $\mathcal{G}'$ and viceversa. We will also show that the reductivity condition does not imply the strong reductivity condition. It is worth pointing out that this situation is not a consequence of the freedom obtained by enlarging the (rather rigid) family of globally homogeneous spaces to the family of locally homogeneous spaces, and we can find illustrative examples restricting ourselves to globally homogeneous pseudo-Riemannian manifolds. We will finally give an example of an stabilizing pair distinct of $(k(p), l(p))$.

Example 3.4.1 *Consider $\mathbb{R}^5$ endowed with the standard metric η of signature $(2,3)$. We take the 4-dimensional submanifold*

$$\mathbb{H}^4_1 = \{x \in \mathbb{R}^5 /\, \eta(x,x) = -1\},$$

endowed with the pseudo-Riemannian metric g inherited from η. $(\mathbb{H}^4_1, g)$ is a Lorentz space of constant sectional curvature, and it is well known that it is the (globally) symmetric space

$$\mathbb{H}^4_1 \simeq \frac{SO_0(2,3)}{SO_0(1,3)}.$$

Let $\{e_1,\ldots,e_5\}$ be the standard basis of $\mathbb{R}^5$, and let e_i^j denote the endomorphism $e^j \otimes e_i$ of $\mathbb{R}^5$. The isotropy algebra at the point $p = (0,1,0,0,0) \in \mathbb{H}^4_1$ is

$$\mathfrak{so}(1,3) = \mathrm{Span}\{e_1^3 + e_3^1, e_1^4 + e_4^1, e_1^5 + e_5^1, e_3^4 - e_4^3, e_3^5 - e_5^3, e_4^5 - e_5^4\}.$$

An $SO_0(1,3)$-invariant complement is

$$\mathfrak{m} = \mathrm{Span}\{e_1^2 - e_2^1, e_2^3 + e_3^2, e_2^4 + e_4^2, e_2^5 + e_5^2\},$$

hence $(\mathbb{H}^4_1, g, SO_0(2,3))$ is reductive. Consider now the Lie subalgebra $\mathfrak{g}$ spanned by the elements

$$e_1^4 + e_4^1 - e_2^3 - e_3^2,\ \frac{1}{2}(e_1^2 - e_2^1 + e_1^3 + e_3^1 + e_2^4 + e_4^2 + e_4^3 - e_3^4),$$

$$\frac{1}{2}(e_1^3 + e_3^1 + e_2^1 - e_1^2 + e_2^4 + e_4^2 + e_3^4 - e_4^3),\ \frac{1}{2}(e_2^1 - e_1^2 + e_2^4 + e_4^2 + e_4^3 - e_3^4 - e_1^3 - e_3^1),$$

$$\frac{1}{\sqrt{2}}(e_1^5 + e_5^1 + e_5^4 - e_4^5),\ \frac{1}{\sqrt{2}}(e_3^5 - e_5^3 - e_2^5 - e_5^2),\ e_1^4 + e_4^1 + e_2^3 + e_3^2.$$

The isotropy algebra $\mathfrak{k}$ at p is spanned by the elements

$$2(e_1^4 + e_4^1 + e_1^4),\ e_1^3 + e_3^1 + e_3^4 - e_4^3,\ \frac{1}{\sqrt{2}}(e_1^5 + e_5^1 + e_5^4 - e_4^5).$$

Let G be the connected Lie subgroup of $SO_0(2,3)$ with Lie algebra $\mathfrak{g}$, then G acts transitively on $\mathbb{H}^4_1$, but there is no $\mathrm{ad}(\mathfrak{k})$-invariant complement of $\mathfrak{k}$, so that $(\mathbb{H}^4_1, g, G)$ is non-reductive (see Lie algebra $A5^*$ in [25]).

Example 3.4.2 *We consider $\mathbb{R}^4$ endowed with the pseudo-Riemannian metric*

$$g = 2e^{y_1}\cos y_2(dy_1dy_4 - dy_2dy_3) - 2e^{y_1}\sin y_2(dy^1dy^3 + dy^2dy^4) + Le^{4y_1}dy_2dy_2,$$

with $L \in \mathbb{R} - \{0\}$. The group $G' = \widetilde{SL(2,\mathbb{R})} \ltimes \mathbb{R}^2 \times \mathbb{R}$ acts transitively by isometries on $(\mathbb{R}^4, g)$ (see §5 of [25]). The Lie algebra of G' can be written as

$$[e_1,e_2] = 2e_2, \qquad [e_1,e_3] = -2e_3, \qquad [e_2,e_3] = e_1, \qquad [e_1,e_4] = e_4,$$

$$[e_1,e_5]=-e_5,\qquad [e_2,e_5]=e_4,\qquad [e_3,e_4]=e_5,$$

with respect to some basis $\{e_1,\ldots,e_6\}$, *which is the Lie algebra B3 in [25]. In addition it is the full isometry algebra of* $(\mathbb{R}^4,g)$ *and can be realized by the complete Killing vector fields*

$$\begin{aligned}
Y_1 &= \cos(2y_2)\partial_{y_1}-\sin(2y_2)\partial_{y_2}+y_3\partial_{y_3}-y_4\partial_{y_4},\\
Y_2 &= \frac{1}{2}\sin(2y_2)\partial_{y_1}+\cos^2(y_2)\partial_{y_2}+y_3\partial_{y_4},\\
Y_3 &= \frac{1}{2}\sin(2y_2)\partial_{y_1}-\sin^2(y_2)\partial_{y_2}+y_4\partial_{y_3},\\
Y_4 &= \partial_{y_4},\\
Y_5 &= -\partial_{y_3},\\
Y_6 &= e^{y_1}\cos(y_2)\partial_{y_3}+e^{y_1}\sin(y_2)\partial_{y_4}.
\end{aligned}$$

The isotropy algebra at $(0,0,0,0)\in\mathbb{R}^4$ *is* $\mathrm{Span}\{e_3,e_5+e_6\}$. *As stated in [25],* $(\mathbb{R}^4,g,G')$ *is non-reductive. We take the subalgebra* $\mathfrak{g}=\mathrm{Span}\{e_1,e_2,e_4,e_5,e_6\}$. *Making use of the distribution generated by the corresponding Killing vector fields we see that the action of the connected Lie subgroup* $G\subset G'$ *with Lie algebra* $\mathfrak{g}$ *is transitive. The isotropy algebra at* $(0,0,0,0)$ *is* $\mathfrak{k}=\mathrm{Span}\{e_5+e_6\}$, *and* $\mathfrak{m}=\mathrm{Span}\{e_1,e_2,e_4,e_5\}$ *is an* $\mathrm{Ad}(K)$*-invariant complement, where* $K\subset G$ *is the isotropy group with respect to the action of* G *at* $(0,0,0,0)$. *Therefore* $(\mathbb{R}^4,g,G)$ *is reductive. On the other hand we can check that* $(\mathbb{R}^4,g)$ *is not strongly reductive. In this case, since there is no extra geometric structure, the complex of filtrations reduces to*

$$\mathfrak{so}(T_pM)\supset\mathfrak{g}(p,0)\supset\mathfrak{g}(p,1)\supset\ldots$$

A simple computation shows that the only non-zero component of the curvature is $R_{\partial_{y_1}\partial_{y_2}\partial_{y_1}\partial_{y_2}}=-3Le^{4y_1}$, *and* $\nabla R=0$. *We take* $p=(0,0,0,0)$ *and* $L=1$ *for the sake of simplicity, so that the filtration actually is*

$$\mathfrak{so}(T_pM)\supset\mathfrak{g}(p,0)=\mathfrak{g}(p,1),$$

where

$$\mathfrak{so}(T_pM)=\left\{\begin{pmatrix} -e & 2(b-c) & b & 0\\ f & 2a & a & c\\ 2(d-f) & 0 & -2a & 2(b-c)\\ 0 & 2(d-f) & d & e\end{pmatrix}/\,a,b,c,d,e,f\in\mathbb{R}\right\},$$

$$\mathfrak{g}(p,0)=\{A\in\mathfrak{so}(T_pM)/\,e=2a,\ f=d\}.$$

It is easy to check that $\mathfrak{g}(p,0)$ *does not admit any complement* $\mathfrak{n}$ *invariant by the adjoint action of* $\mathfrak{g}(p,0)$, *hence* $(\mathbb{R}^4,g)$ *cannot be strongly reductive.*

We finally exhibit an example of a locally homogeneous pseudo-Kähler manifold with an stabilizing pair distinct form (k,l), where as usual (k,l) are the first integers such that $\mathfrak{g}(p,k)=\mathfrak{g}(p,k+1)$ and $\mathfrak{p}(p,l)=\mathfrak{p}(p,l+1)$.

Example 3.4.3 *Consider the space* $\mathbb{C}^2$ *with complex coordinates* (w,z). *We take* $M=\mathbb{C}^2-\{||w||=0\}$ *with the standard complex structure* J *and the pseudo-Riemannian metric*

$$g=dw^1dz^1+dw^2dz^2+b(dw^1dw^1+dw^2dw^2),$$

where $w=w^1+iw^2$, $z=z^1+iz^2$, *and* b *is a function depending on* w^1 *and* w^2 *and satisfying* $\Delta b=\frac{R_0}{||w||^4}$ *for some* $R_0\neq 0$. *This manifold is locally homogeneous* *since it admits an ASK-connection (see Proposition 5.2.7 in Chapter 5). Let* $\theta = -\frac{1}{||w||^2}(w^1dw^1 + w^2dw^2)$*, the curvature tensor and its first covariant derivative are*

$$R = \frac{1}{2}\frac{R_0}{||w||^4}(dw^1 \wedge dw^2 \otimes dw^1 \wedge dw^2), \qquad \nabla R = 4\theta \otimes R.$$

We set $R_0 = 2$ *and take the point* $p = (-1, 0, 0, 0)$*, so that*

$$R_p = dw^1 \wedge dw^2 \otimes dw^1 \wedge dw^2, \qquad \nabla R_p = 4dw^1 \otimes R_p,$$

$$\nabla^2 R_p = (20dw^1 \otimes dw^1 - 4dw^2 \otimes dw^2) \otimes R_p.$$

On the other hand, J_p *is the standard complex structure of* $\mathbb{C}^2$ *and* $\nabla J_p = 0$ *since the manifold is pseudo-Kähler. A straightforward computations thus shows that the complex of filtrations is*

$$\begin{array}{ccccccc}
\mathfrak{so}(\mathbb{R}^4)^6 & \supset & \mathfrak{g}(p,0)^2 & \supset & \mathfrak{g}(p,1)^1 & = & \mathfrak{g}(p,2)^1 \\
\cup & & || & & || & & || \\
\mathfrak{p}(p,0)^4 & \supset & \mathfrak{h}(p,0,0)^2 & \supset & \mathfrak{h}(p,1,0)^1 & = & \mathfrak{h}(p,2,0)^1 \\
|| & & || & & || & & || \\
\mathfrak{p}(p,1)^4 & \supset & \mathfrak{h}(p,0,1)^2 & \supset & \mathfrak{h}(p,1,1)^1 & = & \mathfrak{h}(p,2,1)^1,
\end{array}$$

where superindices indicate dimension. We have that $(k, l) = (1, 0)$*, but* $(r, s) = (1, -1)$ *is a stabilizing pair.*

Chapter 4

Classification of homogeneous structures

In previous sections we have seen how AS-connections and ASK-connections are a useful tool for studying homogeneous and locally homogeneous spaces. For instance, for globally homogeneous manifolds, the presence of an AS-connection (or an ASK-connection) characterizes reductive spaces, and provides a representation as a coset space. For that reason, a classification of the possible AS-connections or ASK-connections can help not only to understand different coset representations of the same homogeneous space, but also to shade light to the structure of the vast world of homogeneous and locally homogeneous spaces. A very efficient way to aboard this problem is to classify the possible homogeneous structures S, which are essentially the torsion of the corresponding AS or ASK-connection. The advantage of this approach is that the work can be completely done at an algebraic level (similar to how intrinsic torsion is studied in Riemannian geometry), and some tools like representation theory may apply (see [26]). In this chapter we show a procedure to classify homogeneous structures associated to AS-connections and ASK-connections whose underlying geometric structure is integrable. We then will apply that procedure to the geometric structures that will be treated in subsequent chapters.

4.1 General procedure

Let S be a homogeneous structure on (M, g) associated to an AS-connection $\widetilde{\nabla}$, we will indistinctively refer by S to the $(1, 2)$-tensor field or the metric equivalent $(0, 3)$-tensor field, that is,

$$S_{XYZ} = g(S_X Y, Z).$$

This convention will be used hereafter. Let ∇ be the Levi-Civita connection of g, equation $\widetilde{\nabla} g = 0$ becomes

$$0 = \widetilde{\nabla} g = \nabla g - S \cdot g = -S \cdot g = 0.$$

This means that for every $X \in \mathfrak{X}(M)$ we have $S_X \cdot g = 0$, where S_X acts as a derivation on the tensor algebra, that is,

$$S_{XYZ} + S_{XZY} = 0, \qquad X, Y, Z \in \mathfrak{X}(M).$$

Let $x \in M$, choosing an orthonormal basis of T_xM we can consider the vector space $V = \mathbb{R}^m$ endowed with the standard symmetric bilinear form $\langle\, ,\, \rangle$ of signature (r, s) as a model of (T_xM, g_x). We take the space of tensors $\mathcal{S}(V) \subset \otimes^3 V^*$ with the same symmetries as the homogeneous structure S, that is

$$\mathcal{S}(V) = \{S \in \otimes^3 V^* /\, S_{XYZ} + S_{XZY} = 0\}.$$

As a vector space $\mathcal{S}(V)$ is isomorphic to $V^* \otimes \wedge^2 V^*$, and carries a non-degenerate symmetric bilinear form defined by

$$\langle S, S' \rangle = \sum_{i,j,k=1}^{m} \varepsilon^i \varepsilon^j \varepsilon^k S_{e_i e_j e_k} S'_{e_i e_j e_k},$$

where $\{e_1, \ldots, e_m\}$ is any othonormal basis of $(V, \langle\, ,\, \rangle)$, and $\varepsilon^i = \langle e_i, e_i \rangle$. Furthermore, there is a natural left action of the orthogonal group $O(r,s)$ given by

$$(A \cdot S)_{XYZ} = S_{A^{-1}XA^{-1}YA^{-1}Z}, \qquad A \in O(V),\ X, Y, Z \in V,$$

turning $\mathcal{S}(V)$ into a $O(r,s)$-module. Identifying $\wedge^2 V^*$ with $\mathfrak{so}(r,s)$, we have that $\mathcal{S}(V) \simeq V^* \otimes \mathfrak{so}(r,s)$, and the action of $O(r,s)$ is seen as the tensor product of the standard representation and the adjoint representation.

Suppose that there is a geometric structure on (M, g) defined by a tensor field P (the case of geometric structures given by more than one tensor field is analogous), and that S is associated to an ASK-connection. Assume moreover that this geometric structure is integrable. Recall that this means that the holonomy of g at $x \in M$ can be seen as a subgroup of the stabilizer H_x of P_x inside $O(T_x M)$, or equivalently that $\nabla P = 0$. Equation $\widetilde{\nabla} P = 0$ thus becomes

$$0 = \nabla P - S \cdot P = -S \cdot P,$$

whence $(S_x)_X$ can be seen as an element of the Lie algebra $\mathfrak{h}_x$ of $H(x)$ for every $X \in T_x M$. With the help of an orthonormal basis of $T_x M$, we consider a tensor field P_0 on V as the model of P_x, and denote by H and $\mathfrak{h}$ the stabilizer of P_0 inside $O(r,s)$ and its Lie algebra respectively. The space of tensors on V with the same symmetries as S is thus identified with $V^* \otimes \mathfrak{h} \subset \mathcal{S}(V)$. The natural action of H as a subgroup of $O(r,s)$ restricts to $V^* \otimes \mathfrak{h}$, as it is just the tensor product of the standard and the adjoint representation. This turns $V^* \otimes \mathfrak{h}$ into an H-module. Note that the action of H is orthogonal with respect to the bilinear form $\langle\, ,\, \rangle$ on $\mathcal{S}(V)$. One of the main differences between the Riemannian and the pseudo-Riemannian cases is that for pseudo-Riemannian metrics $\langle\, ,\, \rangle$ need not be definite. The H-module $V^* \otimes \mathfrak{h}$ can be thus decomposed into the direct sum of mutually orthogonal indecomposable H-submodules. As it happened with the holonomy representation, if g is definite we can assure that $V^* \otimes \mathfrak{h}$ is decomposable into the direct sum of irreducible H-submodules. This is also the case when H is semisimple (for instance all the groups appearing in Berger's list are semisimple).

Assume now that we have decomposed $V^* \otimes \mathfrak{h}$ into H-submodules $W^1, \ldots, W^l$, that is

$$V^* \otimes \mathfrak{h} = W^1 \oplus \ldots \oplus W^l.$$

For every $x \in M$, this gives a decomposition of the H_x-module $T_x^* M \otimes \mathfrak{h}_x$ into H_x-submodules $W_x^1, \ldots, W_x^l$, that is

$$T_x^* M \otimes \mathfrak{h}_x = W_x^1 \oplus \ldots \oplus W_x^l.$$

Proposition 4.1.1 *Let S be a homogeneous structure. If S_x belongs to the submodule W_x^i for some $i = 1, \ldots, l$ at a point $x \in M$, then it belongs to the submodule W_y^i at every point $y \in M$.*

Proof. Since (M, g, P) is at least locally homogeneous with P invariant, there is a linear isometry $\phi : T_x M \to T_y M$ preserving P and S. Taking orthonormal basis we identify $T_x M$ and $T_y M$ with V, so that ϕ turns into an element of the group H. This implies that the induced transformation of $V^* \otimes \mathfrak{h}$ preserves the H-submodules $W^1, \ldots, W^l$. Since S is also invariant by ϕ we have that $S_x \in W_x^i$ if and only if $S_y \in W_y^i$. ■

The previous Proposition shows that a certain decomposition of $V^* \otimes \mathfrak{h}$ into H-submodules induces a classification of homogeneous structures, classification which is thus of purely algebraic nature. That decomposition can be done for every particular case in two ways: using representation theory, and by real tensors. For the representation theory approach we will make use of the techniques in [53], which provide a method to decompose tensor products of representations of Lie groups (see also [5, 7, 64]).

4.2 Some classifications

In this section we apply the classification procedure to some integrable geometric structures. The classifications obtained for each case will lead to the notion of *homogeneous structures of linear type*, which will be a central object of study in the rest of this thesis. During this section all sums of vector spaces are direct sums.

4.2.1 Homogeneous pseudo-Riemannian structures

We consider homogeneous structures S on pseudo-Riemannian manifolds (M, g) without considering any geometric structure on them. In other words we assume that S is associated to just an AS-connection and thus only takes into account (2.1). Such structures will be simply called *homogeneous pseudo-Riemannian structures*. In view of the previous section, in order to classify this kind of homogeneous structures we will decompose the $O(r, s)$-module $\mathcal{S}(V)$ into irreducible submodules. Suppose that $m = \dim M \geq 3$. From the general representation theory for the orthonormal group (see [5, 64]) we have the decomposition

$$V^* \otimes \wedge^2 V^* \simeq V^* \otimes W \otimes \wedge^3 V^*,$$

where W is the irreducible representation of $O(r, s)$ associated to Young element $id + (12) - (23) - (132)$. We now show the explicit expression of the tensors of these submodules. Consider the equivariant map

$$\begin{array}{rccl} c_{12}: & \mathcal{S}(V) & \to & V^* \\ & S & \mapsto & c_{12}(S)(Z) = \sum_{i=1}^m \varepsilon^i S_{e_i e_i Z}, \end{array}$$

where $\{e_i\}$ is any orthonormal basis of V and $\varepsilon^i = \langle e_i, e_i \rangle$. The subspace $ker(c_{12})$ is non-degenerate with respect to the symmetric bilinear form on $\mathcal{S}(V)$, and its orthogonal complement is

$$ker(c_{12})^\perp = \{S \in \mathcal{S}(V) / \, S_{XYZ} = \langle X, Y \rangle \varphi(Z) - \langle X, Z \rangle \varphi(Y), \, \varphi \in V^*\}.$$

We now take the equivariant map

$$\begin{array}{rccl} L: & \mathcal{S}(V) & \to & \mathcal{S}(V) \\ & S & \mapsto & L(S) = \mathfrak{S}_{XYZ} \, S_{XYZ}. \end{array}$$

This map satisfies $L^2 = 3L$ so that it is diagonalizable with real eigenvalues 0 and 3. The corresponding eigenspaces

$$\mathcal{S}^0(V) = \{S \in \mathcal{V} / \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0\}$$
$$\mathcal{S}^3(V) = \{S \in \mathcal{S}(V) / \, S_{XYZ} + S_{YXZ} = 0\}$$

are mutually orthogonal and invariant by $O(r, s)$. It is easy to check that $\mathcal{S}^3(V) \subset ker(c_{12})$ and $ker(c_{12})^\perp \subset \mathcal{S}^0(V)$. We set $\mathcal{S}_1(V) = ker(c_{12})^\perp$, $\mathcal{S}_2(V) = \mathcal{S}^0(V) \cap ker(c_{12})$, and $\mathcal{S}_3(V) = \mathcal{S}^3(V)$.

Proposition 4.2.1 ([60, 31]) *If $m \geq 3$, then the space $\mathcal{S}(V)$ decomposes into irreducible and mutually orthogonal $O(r,s)$-submodules as*

$$\mathcal{S}(V) = \mathcal{S}_1(V) + \mathcal{S}_2(V) + \mathcal{S}_3(V).$$

If $m = 2$, then $\mathcal{S}(V) = \mathcal{S}_1(V)$.

Let $\mathcal{S}$ denote the set of homogeneous pseudo-Riemannian structures, we thus obtain the following classification:

$$\begin{aligned}
\mathcal{S}_1 &= \Big\{ S \in \mathcal{S} \,/\, S_{XYZ} = g(X,Y)\varphi(Z) - g(X,Z)\varphi(Y),\, \varphi \in \Omega^1(M) \Big\} \\
\mathcal{S}_2 &= \Big\{ S \in \mathcal{S} \,/\, \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0,\, c_{12}(S) = 0 \Big\} \\
\mathcal{S}_3 &= \Big\{ S \in \mathcal{S} \,/\, S_{XYZ} + S_{YXZ} = 0 \Big\}
\end{aligned}$$

Taking into account that

$$\dim \mathcal{S}_1(V) = m, \qquad \dim \mathcal{S}_2(V) = \frac{m(m-2)(m+2)}{3}, \qquad \dim \mathcal{S}_3(V) = \binom{m}{3},$$

we note that homogeneous pseudo-Riemannian structures in the class $\mathcal{S}_1$ are sections of a vector bundle whose rank grows linearly with the dimension of the manifold. This motivates the following definition.

Definition 4.2.2 *A homogeneous pseudo-Riemannian structure is called of linear type if it belongs to the class $\mathcal{S}_1$.*

It is easy to see that a homogeneous pseudo-Riemanian structure of linear type seen as a $(1,2)$-tensor field takes the form

$$S_X Y = g(X,Y)\xi - g(Y,\xi)X, \tag{4.1}$$

for some vector field $\xi \in \mathfrak{X}(M)$. In addition, if S takes the previous form then Ambrose-Singer equations are equivalent to

$$\widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} \xi = 0.$$

As we are working with metrics with signature a subclassification of this kind of structures will be needed.

Definition 4.2.3 *A homogeneous pseudo-Riemannian structure of linear type S defined by the vector field ξ is called*

1. *non-degenerate if $g(\xi,\xi) = 0$,*
2. *degenerate if $g(\xi,\xi) = 0$.*

4.2.2 Homogeneous pseudo-Kähler structures

We consider homogeneous structures S on pseudo-Kähler manifolds (M,g,J), satisfying Ambrose-Singer-Kiričenko equtions, that is

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} S = 0, \qquad \widetilde{\nabla} J = 0.$$

Such structures will be called *homogeneous pseudo-Kähler structures*. In this case the stabilizer of the complex structure inside $O(r,s)$ is the corresponding unitary group $U(p,q)$, where $2p = r$ and $2q = s$. We thus decompose the $U(p,q)$-module $\mathcal{K}(V) =$

$V^* \otimes \mathfrak{u}(p,q) \subset \mathcal{S}(V)$ into irreducible submodules. As a representation of $U(p,q)$, the space $\mathcal{K}(V)$ is isomorphic to the tensor product of the standard representation and the adjoint representation of $U(p,q)$. Suppose that $m = \dim M \geq 6$. Following [53] we adopt the notation $[\![\Lambda^{k,l}]\!] \otimes \mathbb{C} = \Lambda^{k,l} + \Lambda^{l,k}$, $[\Lambda^{k,k}] \otimes \mathbb{C} = \Lambda^{k,k}$, where $\Lambda^{k,l}$ denotes the space of forms of type (k,l) on $V \otimes \mathbb{C}$ with respect to J. The standard and the adjoint representation can thus be written as $[\![\Lambda^{0,1}]\!]$ and $\mathbb{R} + [\Lambda_0^{1,1}]$ respectively, where the subindex 0 denotes the primitive part with respect to the symplectic form associated to J. As complex representations we have [26]

$$\Lambda^{1,0} \otimes \Lambda_0^{1,1} \simeq \Lambda^{1,0} + \Lambda_0^{2,1} + S_0^{2,1},$$

where $S_0^{2,1}$ is the kernel of the anti-symmetrization $\Lambda^{1,0} \otimes \Lambda_0^{1,1} \to \Lambda^{2,1}$. We obtain

$$V^* \otimes \mathfrak{u}(p,q) \simeq [\![\Lambda^{0,1}]\!] \otimes \left(\mathbb{R} + [\Lambda_0^{1,1}]\right) \simeq [\![\Lambda^{0,1}]\!] + [\![\Lambda^{0,1}]\!] + [\![\Lambda_0^{2,1}]\!] + [\![S_0^{2,1}]\!].$$

We now give the explicit expression of the tensors in these submodules. Consider the equivariant map

$$\begin{array}{rccl} L: & \mathcal{K}(V) & \to & \mathcal{K}(V) \\ & S & \mapsto & L(S)_{XYZ} = \frac{1}{2}\left(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}\right), \end{array}$$

which is also orthogonal with respect to the symmetric bilinear form inherited from $\mathcal{S}(V)$. As a simple computation shows $L^2 = Id$, so that L is diagonalizable with eigenvalues ± 1. The corresponding eigenspaces $\mathcal{K}^{\pm 1}(V)$ are mutually orthogonal and invariant by $U(p,q)$. Taking the contraction c_{12}, each eigenspace splits in two mutually orthogonal submodules

$$\mathcal{K}^{\pm 1}(V) = \mathcal{K}^{\pm 1}(V) \cap ker(c_{12}) + \mathcal{K}^{\pm 1}(V) \cap ker(c_{12})^{\perp}.$$

We set $\mathcal{K}_1 = \mathcal{K}^1(V) \cap ker(c_{12})$, $\mathcal{K}_2(V) = \mathcal{K}^1(V) \cap ker(c_{12})^{\perp}$, $\mathcal{K}_3(V) = \mathcal{K}^{-1}(V) \cap ker(c_{12})$, and $\mathcal{K}_4(V) = \mathcal{K}^{-1}(V) \cap ker(c_{12})^{\perp}$.

Proposition 4.2.4 ([1, 8]) *If $m \geq 6$, the space $\mathcal{K}(V)$ is decomposed into mutually orthogonal and irreducible $U(p,q)$-submodules as*

$$\mathcal{K}(V) = \mathcal{K}_1(V) + \mathcal{K}_2(V) + \mathcal{K}_3(V) + \mathcal{K}_4(V),$$

where

$$\begin{aligned} \mathcal{K}_1(V) = \Big\{ S \in \mathcal{K}(V) \,/\, S_{XYZ} &= \frac{1}{2}(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}), \\ & \qquad\qquad\qquad\qquad c_{12}(S) = 0 \Big\}, \\ \mathcal{K}_2(V) = \Big\{ S \in \mathcal{K}(V) \,/\, S_{XYZ} &= \langle X,Y\rangle \theta_1(Z) - \langle X,Z\rangle \theta_1(Y) + \langle X,JY\rangle \theta_1(JZ) \\ & \quad - \langle X,JZ\rangle \theta_1(JY) - 2\langle JY,Z\rangle \theta_1(JX),\ \theta_1 \in V^* \Big\}, \\ \mathcal{K}_3(V) = \Big\{ S \in \mathcal{K}(V) \,/\, S_{XYZ} &= -\frac{1}{2}(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}), \\ & \qquad\qquad\qquad\qquad c_{12}(S) = 0 \Big\}, \\ \mathcal{K}_4(V) = \Big\{ S \in \mathcal{K}(V) \,/\, S_{XYZ} &= \langle X,Y\rangle \theta_2(Z) - \langle X,Z\rangle \theta_2(Y) + \langle X,JY\rangle \theta_2(JZ) \\ & \quad - \langle X,JZ\rangle \theta_2(JY) + 2\langle JY,Z\rangle \theta_2(JX),\ \theta_2 \in V^* \Big\}. \end{aligned}$$

If $m = 4$ then $\mathcal{K}(V) = \mathcal{K}_2(V) + \mathcal{K}_3(V) + \mathcal{K}_4(V)$. If $m = 2$ then $\mathcal{K}(V) = \mathcal{K}_4(V)$.

Taking into account that

$$\dim \mathcal{K}_1(V) = n(n+1)(n-2), \qquad \dim \mathcal{K}_2(V) = \dim \mathcal{K}_4(V) = 2n,$$
$$\dim \mathcal{K}_3(V) = n(n-1)(n+2),$$

where $m = 2n$, we note that homogeneous pseudo-Kähler structures in the composed class $\mathcal{K}_2 + \mathcal{K}_4$ are sections of a vector bundle whose rank grows linearly with the dimension of the manifold. This motivates the following definition.

Definition 4.2.5 *A homogeneous pseudo-Kähler structure is called of linear type if it belongs to the class $\mathcal{K}_2 + \mathcal{K}_4$.*

It is easy to see that a homogeneous pseudo-Kähler structure of linear type seen as a $(1,2)$-tensor field takes the form

$$S_X Y = g(X,Y)\xi - g(Y,\xi)X - g(X,JY)J\xi + g(JY,\xi)JX - 2g(JX,\zeta)JY, \qquad (4.2)$$

for some vector fields $\xi, \zeta \in \mathfrak{X}(M)$. In addition, if S takes the previous form, then Ambrose-Singer-Kiričenko equations are equivalent to

$$\widetilde{\nabla} R = 0, \qquad \widetilde{\nabla}\xi = 0, \qquad \widetilde{\nabla}\zeta = 0. \qquad (4.3)$$

As we are working with metrics with signature, a subclassification of this kind of structures will be needed.

Definition 4.2.6 *A homogeneous pseudo-Kähler structure of linear type S defined by the vector fields ξ and ζ is called*

1. *non-degenerate if $g(\xi,\xi) = 0$,*
2. *degenerate if $g(\xi,\xi) = 0$.*
3. *strongly degenerate if $g(\xi,\xi) = 0$ and $\zeta = 0$.*

4.2.3 Homogeneous para-Kähler structures

We consider homogeneous structures S on para-Kähler manifolds (M,g,J), satisfying Ambrose-Singer-Kiričenko equations, that is

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} S = 0, \qquad \widetilde{\nabla} J = 0.$$

Such structures will be called *homogeneous para-Kähler structures*. In this case the stabilizer of the para-complex structure J inside $O(r,s)$ is the para-unitary group $Gl(n,\mathbb{R})$, where $\dim M = m = 2n$. We thus have to decompose the $Gl(n,\mathbb{R})$-module

$$\mathcal{PK}(V) = V^* \otimes \mathfrak{gl}(n,\mathbb{R}) \subset \mathcal{S}(V)$$

into irreducible submodules. Note that V is an indecomposable but reducible representation of $Gl(n,\mathbb{R})$ since there are two invariant maximal isotropic and complementary subspaces of V. These subspaces are exactly the eigenspaces V_+ and V_- corresponding to the eigenvalues ± 1 of J, and they have the same dimension. Taking V^+ and V^- the dual spaces of V_+ and V_- respectively, we have

$$V^* \otimes \mathfrak{gl}(n,\mathbb{R}) \simeq V^+ \otimes \mathfrak{gl}(n,\mathbb{R}) + V^- \otimes \mathfrak{gl}(n,\mathbb{R}).$$

We denote $\Lambda^{k,-l} = \wedge^k V^+ \otimes \wedge^l V^-$. Under the identification $\mathfrak{so}(r,s) \simeq \wedge^2 V^*$ we have that $\mathfrak{gl}(n,\mathbb{R})$ is identified with $\Lambda^{1,-1}$. Let $\omega \in \Lambda^{1,-1}$ be the symplectic form associated to J, we decompose $\Lambda^{1,-1} = \mathbb{R}\omega + \Lambda_0^{1,-1}$. Hence

$$V^+ \otimes \mathfrak{gl}(n,\mathbb{R}) \simeq V^+ \otimes \mathbb{R} + V^+ \otimes \Lambda_0^{1,-1}.$$

On the other hand
$$V^+ \otimes \Lambda_0^{1,-1} \simeq V^+ + \Lambda_0^{2,-1} + S_0^{2,-1},$$
where $S_0^{2,-1}$ is the kernel of the anti-symmetrization $V^+ \otimes \Lambda_0^{1,-1} \to \Lambda^{2,-1}$, and $\Lambda_0^{2,-1}$ is the primitive part of $\Lambda^{2,-1}$ with respect to ω. Since the complexification of $\mathfrak{gl}(n,\mathbb{R})$ is isomorphic to $\mathfrak{gl}(n,\mathbb{C})$, which coincides with the complexification of $\mathfrak{u}(p,q)$, the representation theory for $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{u}(p,q)$ is analogous. Using this fact it is easy to see that the submodules $\Lambda_0^{2,-1}$ and $S_0^{2,-1}$ are irreducible. A decomposition of $V^- \otimes \Lambda^{1,-1}$ is carried out in a similar way. Summarizing we obtain
$$V^* \otimes \mathfrak{gl}(n,\mathbb{R}) \simeq 2V^+ + 2V^- + \Lambda_0^{2,-1} + \Lambda_0^{1,-2} + S_0^{2,-1} + S_0^{1,-2}.$$

On the other hand, we consider the equivariant map
$$\begin{array}{rccl} L: & \mathcal{K}(V) & \to & \mathcal{K}(V) \\ & S & \mapsto & L(S)_{XYZ} = \frac{1}{2}\left(S_{YZX} + S_{ZXY} - S_{JYJZX} - S_{JZXJY}\right), \end{array}$$
which is also orthogonal with respect to the symmetric bilinear form inherited from $\mathcal{S}(V)$. As a simple computation shows $L^2 = Id$, so that L is diagonalizable with eigenvalues ± 1. The corresponding eigenspaces $\mathcal{W}^{\pm 1}(V)$ are mutually orthogonal and invariant by $Gl(n,\mathbb{R})$. Taking now the contraction c_{12}, each eigenspace splits into two mutually orthogonal submodules
$$\mathcal{W}^{\pm 1} = \mathcal{W}^{\pm 1} \cap ker(c_{12}) + \mathcal{W}^{\pm 1} \cap ker(c_{12})^{\perp}.$$
We set $\mathcal{U}_1 = \mathcal{W}^1 \cap ker(c_{12})$, $\mathcal{U}_2 = \mathcal{W}^1 \cap ker(c_{12})^{\perp}$, $\mathcal{U}_3 = \mathcal{W}^{-1} \cap ker(c_{12})$, and $\mathcal{U}_4 = \mathcal{W}^{-1} \cap ker(c_{12})^{\perp}$. The submodules $\mathcal{U}_1, \ldots, \mathcal{U}_4$ are indecomposable but reducible. Indeed, the splitting $V = V_+ + V_-$ induce decompositions $\mathcal{U}_i = \mathcal{U}_i^+ + \mathcal{U}_i^-$ where
$$\mathcal{U}_i^+ = \left\{ S \in \mathcal{U}_i / \, S_{X_- YZ} = 0, \; X_- \in V_- \right\},$$
$$\mathcal{U}_i^- = \left\{ S \in \mathcal{U}_i / \, S_{X_+ YZ} = 0, \; X_+ \in V_+ \right\}.$$
We set
$$\mathcal{PK}_1(V) = \mathcal{U}_1^+, \; \mathcal{PK}_2(V) = \mathcal{U}_2^+, \; \mathcal{PK}_3(V) = \mathcal{U}_3^+, \; \mathcal{PK}_4(V) = \mathcal{U}_4^+,$$
$$\mathcal{PK}_5(V) = \mathcal{U}_1^-, \; \mathcal{PK}_6(V) = \mathcal{U}_2^-, \; \mathcal{PK}_7(V) = \mathcal{U}_3^-, \; \mathcal{PK}_8(V) = \mathcal{U}_4^-.$$

Proposition 4.2.7 ([30]) *If $m \geq 6$, the space $\mathcal{PK}(V)$ is decomposed into irreducible $Gl(n,\mathbb{R})$-submodules as*
$$\begin{aligned} \mathcal{PK}(V) = {} & \mathcal{PK}_1(V) + \mathcal{PK}_2(V) + \mathcal{PK}_3(V) + \mathcal{PK}_4(V) \\ & + \mathcal{PK}_5(V) + \mathcal{PK}_6(V) + \mathcal{PK}_7(V) + \mathcal{PK}_8(V), \end{aligned}$$
where
$$\begin{aligned} \mathcal{PK}_1(V) = \Big\{ S \in \mathcal{PK}(V) \, / \, S_{XYZ} = {} & \frac{1}{2}(S_{YZX} + S_{ZXY} - S_{JYJZX} - S_{JZXJY}), \\ & c_{12}(S) = 0, \; S_{X_- YZ} = 0, \; X_- \in V_- \Big\}, \end{aligned}$$
$$\begin{aligned} \mathcal{PK}_2(V) = \Big\{ S \in \mathcal{PK}(V) \, / \, S_{XYZ} = {} & \langle X, Y \rangle \theta_1(Z) - \langle X, Z \rangle \theta_1(Y) - \langle X, JY \rangle \theta_1(JZ) \\ & + \langle X, JZ \rangle \theta_1(JY) + 2\langle JY, Z \rangle \theta_1(JX), \; \theta_1 \in V^+ \Big\}, \end{aligned}$$
$$\begin{aligned} \mathcal{PK}_3(V) = \Big\{ S \in \mathcal{PK}(V) \, / \, S_{XYZ} = {} & -\frac{1}{2}(S_{YZX} + S_{ZXY} - S_{JYJZX} - S_{JZXJY}), \\ & c_{12}(S) = 0, \; S_{X_- YZ} = 0, \; X_- \in V_- \Big\}, \end{aligned}$$
$$\begin{aligned} \mathcal{PK}_4(V) = \Big\{ S \in \mathcal{PK}(V) \, / \, S_{XYZ} = {} & \langle X, Y \rangle \theta_2(Z) - \langle X, Z \rangle \theta_2(Y) - \langle X, JY \rangle \theta_2(JZ) \\ & + \langle X, JZ \rangle \theta_2(JY) - 2\langle JY, Z \rangle \theta_2(JX), \; \theta_2 \in V^+ \Big\}, \end{aligned}$$

and $\mathcal{PK}_5, \ldots, \mathcal{PK}_8$ *are obtained from* $\mathcal{PK}_1, \ldots, \mathcal{PK}_4$ *interchanging* $+$ *by* $-$*. If* $m = 4$ *then*

$$\mathcal{PK}(V) = \mathcal{PK}_2(V) + \mathcal{PK}_3(V) + \mathcal{K}_4(V) + \mathcal{PK}_6(V) + \mathcal{PK}_7(V) + \mathcal{K}_8(V).$$

If $m = 2$ *then* $\mathcal{K}(V) = \mathcal{PK}_4(V) + \mathcal{PK}_8(V)$*.*

Taking into account that

$$\dim \mathcal{PK}_1(V) = \dim \mathcal{PK}_5(V) = \frac{n(n+1)(n-2)}{2},$$

$$\dim \mathcal{PK}_3(V) = \dim \mathcal{PK}_7(V) = \frac{n^2(n+1)}{2} - n,$$

$$\dim \mathcal{PK}_2(V) = \dim \mathcal{PK}_4(V) = \dim \mathcal{PK}_6(V) = \dim \mathcal{PK}_8(V) = n,$$

we note that homogeneous para-Kähler structures in the composed class $\mathcal{PK}_2 + \mathcal{PK}_4 + \mathcal{PK}_6 + \mathcal{PK}_8$ are sections of a vector bundle whose rank grows linearly with the dimension of the manifold. This motivates the following definition.

Definition 4.2.8 *A homogeneous para-Kähler structure is called of linear type if it belongs to the class* $\mathcal{PK}_2 + \mathcal{PK}_4 + \mathcal{PK}_6 + \mathcal{PK}_8$*.*

It is easy to see that a homogeneous para-Kähler structure of linear type seen as a $(1,2)$-tensor field takes the form

$$S_X Y = g(X,Y)\xi - g(Y,\xi)X + g(X,JY)J\xi - g(JY,\xi)JX - 2g(JX,\zeta)JY, \tag{4.4}$$

for some vector fields $\xi, \zeta \in \mathfrak{X}(M)$. In addition, if S takes the previous form, then Ambrose-Singer-Kiričenko equations are equivalent to

$$\widetilde{\nabla} R = 0, \qquad \widetilde{\nabla}\xi = 0, \qquad \widetilde{\nabla}\zeta = 0. \tag{4.5}$$

As the underlying metric g is not definite, a subclassification of this kind of structures will be needed.

Definition 4.2.9 *A homogeneous para-Kähler structure of linear type* S *defined by the vector fields* ξ *and* ζ *is called*

1. *non-degenerate if* $g(\xi,\xi) = 0$*,*
2. *degenerate if* $g(\xi,\xi) = 0$*.*
3. *strongly degenerate if* $g(\xi,\xi) = 0$ *and* $\zeta = 0$*.*

4.2.4 Homogeneous pseudo-quaternion Kähler structures

Let (M, g, Q) be a pseudo-quaternion Kähler manifold, we consider homogeneous structures S satisfying Ambrose-Singer-Kiričenko equations, that is

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} S = 0, \qquad \widetilde{\nabla}\Omega = 0.$$

Such structures will be called *homogeneous pseudo-quaternion Kähler structures.* In this case, the stabilizer of the 4-form Ω inside $O(r,s)$ is $Sp(p,q)Sp(1)$, where $4p = r$ and $4q = s$. We thus decompose the $Sp(p,q)Sp(1)$-module $\mathcal{QK}(V) = V^* \otimes (\mathfrak{sp}(p,q) + \mathfrak{sp}(1)) \subset \mathcal{S}(V)$ into irreducible submodules. Suppose that $m = \dim M \geq 8$. Let $n = p + q$, following [16] we denote by $E = \mathbb{C}^{2n}$ and $H = \mathbb{C}^2$ the standard representations of $Sp(p,q)$ and $Sp(1)$ respectively. The product of the quaternionic structures on E and H gives real structures on tensor products of E and H, and we denote the real part with respect to these real structures with brackets. For the sake of simplicity we will omit some tensor products and write EH for $E \otimes H$, etc. The standard representation of $Sp(p,q)Sp(1)$ is thus $V = [EH]$, and the adjoint representation is isomorphic to $[S^2E] + [S^2H]$. We thus have

$$V^* \otimes (\mathfrak{sp}(p,q) + \mathfrak{sp}(1)) \simeq [EH] \otimes ([S^2E] + [S^2H]) \simeq [EH] \otimes [S^2E] + [EH] \otimes [S^2H].$$

As complex representations E and S^2E has highest weights $(1,0,\dots,0)$ and $(2,0,\dots,0)$ respectively. Making use of the techniques in [53] we have the decomposition

$$\begin{aligned}(1,0,\dots,0) \otimes (2,0,\dots,0) &\simeq (2,1,0\dots,0) + (3,0,\dots,0) + (1,0,\dots,0)\\ &\simeq K + S^3E + E,\end{aligned}$$

where K is the module associated to the irreducible representation with highest weight $(2,1,0,\dots,0)$. In addition, $H \otimes S^2H \simeq H + S^3H$, where H is identified with the kernel of the symmetrization $H \otimes S^2H \to S^3H$. Summarizing we have the decomposition into irreducible representations [26]

$$V^* \otimes (\mathfrak{sp}(p,q) + \mathfrak{sp}(1)) = [EH] + [ES^2H] + [EH] + [S^3EH] + [KH].$$

We now show the explicit espression of the tensors. The condition $\widetilde{\nabla}\Omega$ is equivalent to

$$\widetilde{\nabla}_X J_a = \sum_{b=1}^{3} d_{ab} J_b, \qquad a = 1,2,3, \tag{4.6}$$

where (d_{ab}) is a matrix of 1-forms sitting in $\mathfrak{sp}(1)$. This implies that

$$J_a(S_XY) - S_X(J_aY) = \sum_{b=1}^{3} c_{ab}(X) J_b Y, \qquad a = 1,2,3,$$

where (c_{ab}) a matrix of 1-forms sitting in $\mathfrak{sp}(1)$. Note that (c_{ab}) can be obtained as the $\mathfrak{sp}(1)$-part of $S_X \in \mathfrak{sp}(p,q) + \mathfrak{sp}(1)$. The symmetries satisfied by a pseudo-quaternion Kähler structure are thus

$$S_{XYZ} = -\, S_{XZY}, \tag{4.7}$$

$$S_{XJ_aYJ_aZ} - S_{X,Y,Z} = -\,\pi^c(X) g(J_bY, J_aZ) + \pi^b(X) g(J_cY, J_aZ), \tag{4.8}$$

for any cyclic permutation (a,b,c) of $(1,2,3)$, where π^1, π^2, π^3 are local 1-forms on M, and Einstein summation convention is used. We take the sum of the three equations in (4.8), which up to a factor 4 is $\Theta^S = \frac{1}{2}\pi^a \otimes \omega_a$. Using the left hand sides of (4.8) one sees that Θ^S satisfies (4.7) and (4.8) for the same 1-forms π^1, π^2, π^3 as S. We now consider the tensor $T^S_{XYZ} = \frac{1}{4}\left(S_{XYZ} + \sum_{a=1}^{3} S_{XJ_aYJ_aZ}\right)$. This tensor satisfies $T^S_{XJ_aYJ_aZ} - T^S_{XYZ} = 0$, $a = 1,2,3$, and $T^S + \Theta^S = S$. We can thus define invariant subspaces of $\mathcal{QK}(V)$

$$\begin{aligned}\check{\mathcal{V}} &= \left\{\Theta \in \mathcal{QK}(V) /\ \Theta_{XYZ} = \frac{1}{2}\pi^a(X)\langle J_aY, Z\rangle,\ \pi^a \in V^*,\ a = 1,2,3\right\}\\ \hat{\mathcal{V}} &= \left\{T \in \mathcal{QK}(V) /\ T^S_{XJ_aYJ_aZ} - T^S_{XYZ} = 0,\ a = 1,2,3\right\},\end{aligned}$$

so that

$$\mathcal{QK}(V) = \check{\mathcal{V}} + \hat{\mathcal{V}}.$$

The kernel of the equivariant map c_{12} restricted to $\check{\mathcal{V}}$ gives the space $\check{\mathcal{V}}_0 = ker(c_{12}) \subset \check{\mathcal{V}}$, on which the symmetric bilinear form inherited from $\mathcal{S}(\mathcal{V})$ is non-degenerate. Hence $\check{\mathcal{V}} = \check{\mathcal{V}}_0 + \check{\mathcal{V}}_0^\perp$, where

$$\check{\mathcal{V}}_0^\perp = \left\{\Theta \in \check{\mathcal{V}} /\ \Theta = -\sum_a (\theta \circ J_a) \otimes \omega_a,\ \theta \in V^*\right\}.$$

Regarding the space $\hat{\mathcal{V}}$, we consider the equivariant map

$$\begin{array}{rccl} L: & \hat{\mathcal{V}} & \to & \hat{\mathcal{V}} \\ & S & \mapsto & L(S)_{XYZ} = S_{ZXY} + S_{YZX} + \sum_{a=1}^{3} \left(S_{J_aYJ_aZX} + S_{J_aZXJ_aY} \right). \end{array}$$

This map satisfies $L^2 = 8Id - 2L$, so that it is diagonalizable with eigenvalues 2 and -4. Denoting by $\hat{\mathcal{V}}^2$ and $\hat{\mathcal{V}}^{-4}$ the corresponding mutually orthogonal eigenspaces we have $\hat{\mathcal{V}} = \hat{\mathcal{V}}^2 + \hat{\mathcal{V}}^{-4}$. The kernel $\hat{\mathcal{V}}_0 \subset \hat{\mathcal{V}}$ of the restriction of c_{12} to $\hat{\mathcal{V}}$ is non-degenerate with respect to the inherited symmetric bilinear form, so that we can consider its orthogonal complement $\hat{\mathcal{V}}_0^\perp \subset \hat{\mathcal{V}}$. A simple inspection shows that $\hat{\mathcal{V}}^{-4} \subset \hat{\mathcal{V}}_0$ and $\hat{\mathcal{V}}_0^\perp \subset \hat{\mathcal{V}}_2$, whence we conclude

$$\mathcal{QK}(V) = \check{\mathcal{V}}_0 + \check{\mathcal{V}}_0^\perp + \hat{\mathcal{V}}_0^\perp + (\hat{\mathcal{V}}^2 \cap \hat{\mathcal{V}}_0) + \hat{\mathcal{V}}^{-4}.$$

We set $\mathcal{QK}_1(V) = \check{\mathcal{V}}_0$, $\mathcal{QK}_2(V) = \check{\mathcal{V}}_0^\perp$, $\mathcal{QK}_3(V) = \hat{\mathcal{V}}_0^\perp$, $\mathcal{QK}_4(V) = \hat{\mathcal{V}}^2 \cap \hat{\mathcal{V}}_0$, and $\mathcal{QK}_5(V) = \hat{\mathcal{V}}^{-4}$.

Proposition 4.2.10 ([8, 16]) *For $m \geq 8$, the space $\mathcal{QK}(V)$ decomposes into irreducible and mutually orthogonal $Sp(p,q)Sp(1)$-submodules as*

$$\mathcal{QK}(V) = \mathcal{QK}_1(V) + \mathcal{QK}_2(V) + \mathcal{QK}_3(V) + \mathcal{QK}_4(V) + \mathcal{QK}_5(V),$$

where

$$\begin{aligned}
\mathcal{QK}_1(V) &= \Big\{ S \in \mathcal{QK}(V) / \, S_{XYZ} = \sum_{a=1}^{3} \theta(J_aX)\langle J_aY, Z\rangle, \ \theta \in V^* \Big\}, \\
\mathcal{QK}_2(V) &= \Big\{ S \in \mathcal{QK}(V) / \, S_{XYZ} = \sum_{a=1}^{3} \theta^a(X)\langle J_aY, Z\rangle, \sum_{a=1}^{3} \theta^a \circ J_a = 0, \theta^a \in V^* \Big\}, \\
\mathcal{QK}_3(V) &= \Big\{ S \in \mathcal{QK}(V) / \, S_{XYZ} = \langle X, Y\rangle\theta(Z) - \langle X, Z\rangle\theta(Y) \\
&\qquad + \sum_{a=1}^{3} (\langle X, J_aY\rangle\theta(J_aZ) - \langle X, J_aZ\rangle\theta(J_aY)), \ \theta \in V^* \Big\}, \\
\mathcal{QK}_4(V) &= \Big\{ S \in \mathcal{QK}(V) / \, 6S_{XYZ} = \mathop{\mathfrak{S}}_{XYZ} S_{XYZ} + \sum_{a=1}^{3} \mathop{\mathfrak{S}}_{XJ_aYJ_aZ} S_{XJ_aYJ_aZ}, \\
&\qquad\qquad c_{12}(S) = 0 \Big\}, \\
\mathcal{QK}_5(V) &= \Big\{ S \in \mathcal{QK}(V) / \mathop{\mathfrak{S}}_{XYZ} S_{XYZ} = 0 \Big\}.
\end{aligned}$$

If $m = 4$ then $\mathcal{QK}(V) = \mathcal{QK}_1(V) + \mathcal{QK}_2(V) + \mathcal{QK}_3(V) + \mathcal{QK}_4(V)$.

Making use of the isomorphisms $\mathcal{QK}_1(V) \simeq \mathcal{QK}_3(V) \simeq [EH]$, $\mathcal{QK}_2(V) = [ES^3H]$, $\mathcal{QK}_4(V) \simeq [S^3EH]$, and $\mathcal{QK}_5(V) \simeq [KH]$, we find

$$\dim \mathcal{QK}_1(V) = \dim \mathcal{QK}_3(V) = 4n, \qquad \dim \mathcal{QK}_2(V) = 8n,$$

$$\dim \mathcal{QK}_4(V) = \frac{4}{3}n(n+1)(2n+1), \qquad \dim \mathcal{QK}_5(V) = \frac{16}{3}n(n^2-1).$$

Therefore, pseudo-quaternion Kähler structures in the composed class $\mathcal{QK}_1 + \mathcal{QK}_2 + \mathcal{QK}_3$ are sections of a vector bundle whose rank grows linearly with the dimension of the manifold. This motivates the following definition.

Definition 4.2.11 *A homogeneous pseudo-quaternion Kähler structure is called of linear type if it belongs to the class $\mathcal{QK}_1 + \mathcal{QK}_2 + \mathcal{QK}_3$.*

It is easy to see that a homogeneous pseudo-quaternion Kähler structure of linear type takes the form

$$S_X Y = g(X,Y)\xi - g(Y,\xi)X + \sum_{a=1}^{3} (g(J_a Y,\xi)J_a X - g(X,J_a Y)J_a \xi) + \sum_{a=1}^{3} g(X,\zeta^a)J_a Y, \quad (4.9)$$

for some vector fields ξ and ζ^a, $a = 1,2,3$. When working with metrics with signature we need the following further definition.

Definition 4.2.12 *A homogeneous pseudo-quaternion Kähler structure of linear type defined by the vector fields ξ and ζ^a, $a = 1,2,3$, is called*

1. *non-degenerate if $g(\xi,\xi) = 0$,*
2. *degenerate if $g(\xi,\xi) = 0$.*

A *homogeneous pseudo-hyper-Kähler structure* is a homogeneous structure on a pseudo-hyper-Kähler manifold (M,g,J_1,J_2,J_3) satisfying $\widetilde{\nabla} J_a = 0$, $a = 1,2,3$. This implies that S satisfies (4.7) and (4.8) with $\pi^a = 0$ for $a = 1,2,3$. Therefore, the decomposition of $\mathcal{HK}(V) = V^* \otimes \mathfrak{sp}(r,s)$ can be read in terms of the pseudo-quaternion Kähler case. In fact,

$$\mathcal{HK}(V) = \mathcal{HK}_1(V) + \mathcal{HK}_2(V) + \mathcal{HK}_3(V),$$

where $\mathcal{HK}_1(V)$, $\mathcal{HK}_2(V)$, $\mathcal{HK}_3(V)$ have the same expression as $\mathcal{QK}_3(V)$, $\mathcal{QK}_4(V)$ and $\mathcal{QK}_5(V)$ respectively. A homogeneous pseudo-hyper-Kähler structure S is said of *linear type* if it belongs to the class $\mathcal{HK}_1$. In that case, S seen as a $(1,2)$-tensor field has the form

$$S_X Y = g(X,Y)\xi - g(Y,\xi)X + \sum_{a=1}^{3} (g(J_a Y,\xi)J_a X - g(X,J_a Y)J_a \xi),$$

where $\xi = \theta^\sharp$. In addition, $S \in \mathcal{HK}_1(V)$ is said degenerate if ξ is isotropic, and non-degenerate if ξ is non-isotropic.

4.2.5 Homogeneous para-quaternion Kähler structures

We now develop the classification of homogeneous para-quaternion Kähler structures, which is not found in the literature. Many of the arguments can be adapted from the pseudo-quaternion Kähler case.

Let (M,g,Q) be a para-quaternion Kähler manifold, we consider homogeneous structures S satisfying Ambrose-Singer-Kiričenko equations, that is

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} S = 0, \qquad \widetilde{\nabla} \Omega = 0.$$

Such structures will be called *homogeneous para-quaternion Kähler structures*. In this case the stabilizer of the 4-form Ω inside $O(r,s)$ is $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$, where $\dim M = m = 4n \geq 8$. We thus decompose the $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$-module $\mathcal{PQ}(V) = V^* \otimes (\mathfrak{sp}(n,\mathbb{R}) + \mathfrak{sp}(1,\mathbb{R})) \subset \mathcal{S}(V)$ into irreducible submodules. We denote by $E = \mathbb{R}^{2n}$ and $H = \mathbb{R}^2$ the standard representations of $Sp(n,\mathbb{R})$ and $Sp(1,\mathbb{R})$ respectively. Fort he sake of simplicity we will omit some tensor products and write EH for $E \otimes H$, etc. The standard representation of $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$ is thus $V = EH$, and the adjoint representation is isomorphic to $S^2E + S^2H$. We thus have

$$V^* \otimes (\mathfrak{sp}(n,\mathbb{R}) + \mathfrak{sp}(1,\mathbb{R})) \simeq EH \otimes (S^2E + S^2H) \simeq EH \otimes S^2E + EH \otimes S^2H.$$

The representations E and S^2E has highest weights $(1,0,\dots,0)$ and $(2,0,\dots,0)$ respectively. Making use of the techniques in [53] and the fact that the complexifications of $\mathfrak{sp}(n,\mathbb{R})$ and $\mathfrak{sp}(n)$ coincide, we have the decomposition

$$\begin{aligned}(1,0,\dots,0)\otimes(2,0,\dots,0) &\simeq (2,1,0\dots,0)+(3,0,\dots,0)+(1,0,\dots,0)\\ &\simeq K+S^3E+E,\end{aligned}$$

where K is the module associated to the irreducible representation with highest weight $(2,1,0,\dots,0)$. In addition, $H\otimes S^2H\simeq H+S^3H$, where H is identified with the kernel of the symmetrization $H\otimes S^2H\to S^3H$. Summarizing we have the decomposition into irreducible representations

$$V^*\otimes(\mathfrak{sp}(n,\mathbb{R})+\mathfrak{sp}(1,\mathbb{R}))=EH+ES^2H+EH+S^3EH+KH. \tag{4.10}$$

Regarding real tensors, we introduce the notation $(\epsilon_1,\epsilon_2,\epsilon_3)=(-1,1,1)$, so that $J_a^2=\epsilon_a$ for $a=1,2,3$. The condition $\widetilde{\nabla}\Omega$ is equivalent to

$$\widetilde{\nabla}_X J_a=\sum_{b=1}^{3} d_{ab}J_b,\qquad a=1,2,3, \tag{4.11}$$

where (d_{ab}) is a matrix of 1-forms sitting in $\mathfrak{sp}(1,\mathbb{R})$. This implies that

$$J_a(S_XY)-S_X(J_aY)=\sum_{b=1}^{3} c_{ab}(X)J_bY,\qquad a=1,2,3,$$

where (c_{ab}) a matrix of 1-forms sitting in $\mathfrak{sp}(1,\mathbb{R})$. Note that (c_{ab}) can be obtained as the $\mathfrak{sp}(1,\mathbb{R})$-part of $S_X\in\mathfrak{sp}(n,\mathbb{R})+\mathfrak{sp}(1,\mathbb{R})$. The symmetries satisfied by a para-quaternion Kähler structure are thus

$$S_{XYZ}=-\,S_{XZY}, \tag{4.12}$$

$$S_{XJ_aYJ_aZ}-S_{X,Y,Z}=\epsilon_b\pi^c(X)g(J_bY,J_aZ)-\epsilon_c\pi^b(X)g(J_cY,J_aZ), \tag{4.13}$$

for any cyclic permutation (a,b,c) of $(1,2,3)$, where π^1,π^2,π^3 are local 1-forms on M, and Einstein summation convention is used. We take the sum of the three equations in (4.8) but each one multiplied by ϵ_a. Up to a factor 4 this gives $\Theta^S=\frac{1}{2}\pi^a\otimes\omega_a$. Using the left hand sides of (4.8) one sees that Θ^S satisfies (4.7) and (4.8) for the same 1-forms π^1,π^2,π^3 as S. We now consider the tensor $T^S_{XYZ}=\frac{1}{4}\left(S_{XYZ}-\sum_{a=1}^{3}\epsilon_aS_{XJ_aYJ_aZ}\right)$. This tensor satisfies $T^S_{XJ_aYJ_aZ}+\epsilon_aT^S_{XYZ}=0$, $a=1,2,3$, and $T^S+\Theta^S=S$. We can thus define invariant subspaces of $\mathcal{PQ}(V)$

$$\begin{aligned}\check{\mathcal{V}}&=\left\{\Theta\in\mathcal{PQ}(V)/\ \Theta_{XYZ}=\frac{1}{2}\pi^a(X)\langle J_aY,Z\rangle,\ \pi^a\in V^*,\ a=1,2,3\right\}\\ \hat{\mathcal{V}}&=\left\{T\in\mathcal{PQ}(V)/\ T^S_{XJ_aYJ_aZ}+\epsilon_aT^S_{XYZ}=0,\ a=1,2,3\right\},\end{aligned}$$

so that

$$\mathcal{PQ}(V)=\check{\mathcal{V}}+\hat{\mathcal{V}}.$$

The kernel of the equivariant map c_{12} restricted to $\check{\mathcal{V}}$ gives the space $\check{\mathcal{V}}_0=ker(c_{12})\subset\check{\mathcal{V}}$, on which the symmetric bilinear form inherited from $\mathcal{S}(\mathcal{V})$ is non-degenerate. Hence $\check{\mathcal{V}}=\check{\mathcal{V}}_0+\check{\mathcal{V}}_0^\perp$, where

$$\check{\mathcal{V}}_0^\perp=\left\{\Theta\in\check{\mathcal{V}}/\ \Theta=-\sum_a(\theta\circ J_a)\otimes\omega_a,\ \theta\in V^*\right\}.$$

Regarding the space $\hat{\mathcal{V}}$, we consider the self-adjoint equivariant map

$$\begin{array}{rccl} L: & \hat{\mathcal{V}} & \to & \hat{\mathcal{V}} \\ & S & \mapsto & L(S)_{XYZ} = S_{ZXY} + S_{YZX} - \sum_{a=1}^3 \epsilon_a \left(S_{J_aYJ_aZX} + S_{J_aZXJ_aY}\right). \end{array}$$

This map satisfies $L^2 = 8Id - 2L$, so that it is diagonalizable with eigenvalues 2 and -4. Denoting by $\hat{\mathcal{V}}^2$ and $\hat{\mathcal{V}}^{-4}$ the corresponding mutually orthogonal eigenspaces we have $\hat{\mathcal{V}} = \hat{\mathcal{V}}^2 + \hat{\mathcal{V}}^{-4}$. The kernel $\hat{\mathcal{V}}_0 \subset \hat{\mathcal{V}}$ of the restriction of c_{12} to $\hat{\mathcal{V}}$ is non-degenerate with respect to the inherited symmetric bilinear form, so that we can consider its orthogonal complement $\hat{\mathcal{V}}_0^{\perp} \subset \hat{\mathcal{V}}$. A simple inspection shows that $\hat{\mathcal{V}}^{-4} \subset \hat{\mathcal{V}}_0$ and $\hat{\mathcal{V}}_0^{\perp} \subset \hat{\mathcal{V}}_2$, whence we conclude

$$\mathcal{PQ}(V) = \check{\mathcal{V}}_0 + \check{\mathcal{V}}_0^{\perp} + \hat{\mathcal{V}}_0^{\perp} + (\hat{\mathcal{V}}^2 \cap \hat{\mathcal{V}}_0) + \hat{\mathcal{V}}^{-4}.$$

We set $\mathcal{PQ}_1(V) = \check{\mathcal{V}}_0$, $\mathcal{PQ}_2(V) = \check{\mathcal{V}}_0^{\perp}$, $\mathcal{PQ}_3(V) = \hat{\mathcal{V}}_0^{\perp}$, $\mathcal{PQ}_4(V) = \hat{\mathcal{V}}^2 \cap \hat{\mathcal{V}}_0$, and $\mathcal{PQ}_5(V) = \hat{\mathcal{V}}^{-4}$.

Proposition 4.2.13 *For $m \geq 8$, the space $\mathcal{PQ}(V)$ decomposes into irreducible and mutually orthogonal $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$-submodules as*

$$\mathcal{PQ}(V) = \mathcal{PQ}_1(V) + \mathcal{PQ}_2(V) + \mathcal{PQ}_3(V) + \mathcal{PQ}_4(V) + \mathcal{PQ}_5(V),$$

where

$$\mathcal{PQ}_1(V) = \Big\{S \in \mathcal{PQ}(V)/\, S_{XYZ} = \sum_{a=1}^3 \theta(J_aX)\langle J_aY, Z\rangle,\ \theta \in V^*\Big\},$$

$$\mathcal{PQ}_2(V) = \Big\{S \in \mathcal{PQ}(V)/\, S_{XYZ} = \sum_{a=1}^3 \theta^a(X)\langle J_aY, Z\rangle, \sum_{a=1}^3 \theta^a \circ J_a = 0, \theta^a \in V^*\Big\},$$

$$\begin{aligned}\mathcal{PQ}_3(V) = \Big\{S \in \mathcal{PQ}(V)/\, S_{XYZ} &= \langle X, Y\rangle\theta(Z) - \langle X, Z\rangle\theta(Y) \\ &- \sum_{a=1}^3 \epsilon_a(\langle X, J_aY\rangle\theta(J_aZ) - \langle X, J_aZ\rangle\theta(J_aY)),\, \theta \in V^*\Big\},\end{aligned}$$

$$\begin{aligned}\mathcal{PQ}_4(V) = \Big\{S \in \mathcal{PQ}(V)/\, 6S_{XYZ} = \underset{XYZ}{\mathfrak{S}} S_{XYZ} - \sum_{a=1}^3 \epsilon_a \underset{XJ_aYJ_aZ}{\mathfrak{S}} S_{XJ_aYJ_aZ}, \\ c_{12}(S) = 0\Big\},\end{aligned}$$

$$\mathcal{PQ}_5(V) = \Big\{S \in \mathcal{PQ}(V)/ \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0\Big\}.$$

If $m = 4$ then $\mathcal{PQ}(V) = \mathcal{PQ}_1(V) + \mathcal{PQ}_2(V) + \mathcal{PQ}_3(V) + \mathcal{PQ}_4(V)$.

Proof. This proof is a straightforward adaptation of the proof for the quaternion Kähler case appearing in [16]. We shall identify each submodule $\mathcal{PQ}_i(V)$ with one appearing in decomposition (4.10). First we obviously have $\mathcal{PQ}_1(V) \simeq EH \simeq \mathcal{PQ}_3(V)$ with non-zero projections

$$EH \otimes S^2H \to \mathcal{PQ}_1(V), \qquad EH \otimes S^2E \to \mathcal{PQ}_3(V).$$

On the other hand, as $\dim \mathcal{PQ}_2(V) = 8n$ and $\mathcal{PQ}_2(V)$ is orthogonal to $\mathcal{PQ}_1(V)$ inside $\check{\mathcal{V}}$, we have that $\mathcal{PQ}_2(V) \simeq ES^3H$. Now, $\mathcal{PQ}_3(V) + \mathcal{PQ}_4(V) + \mathcal{PQ}_5(V) \simeq EH + S^3EH + KH$. Observe that

$$\wedge^2 E \otimes E = \mathbb{R}\omega \otimes E + (\mathbb{R}\omega)^{\perp} \otimes E \simeq E + \wedge_0^2 E \otimes E \simeq 2E + K + V^{(1,1,1,0,\ldots,0)},$$

where $V^{(1,1,1,0,\ldots,0)}$ is the representation with highest weight $(1,1,1,0,\ldots,0)$. Hence KH is a submodule of both

$$S^2V^* \otimes V^* \supset \wedge^2 E \otimes EH, \qquad V^* \otimes \wedge^2 E \supset EH \otimes S^2 E.$$

Using Schur's Lemma, an equivariant map $\mathcal{PQ}_4(V) + \mathcal{PQ}_5(V) \to \wedge^2 E \otimes EH$ will have kernel isomorphic to $S^3 EH$ and will be non-zero in a submodule isomorphic to K. The module $\wedge^2 E \subset S^2 V^*$ can be seen as the space of bilinear forms satisfying $b(J_a\cdot, J_a\cdot) = \epsilon_a b(\cdot,\cdot)$, $a = 1, 2, 3$, so that we consider

$$\begin{array}{rrcl} p: & S^2V^* \otimes V^* & \to & \wedge^2 E \otimes EH \\ & T & \mapsto & \frac{1}{4}\left(T_{XYZ} - \sum_a \epsilon_a T_{J_aXJ_aYZ}\right). \end{array}$$

The projection $\pi : V^* \otimes \wedge^2 V^* \to \mathcal{PQ}_5(V)$ is given by $T \mapsto \frac{1}{6}(2 - L)U$ where

$$U_{XYZ} = \frac{1}{4}\left(T_{XYZ} - \sum_a \epsilon_a T_{XJ_aYJ_aZ}\right).$$

After a long computation one can check that the image under the composition of maps

$$V^* \otimes \wedge^2 V^* \xrightarrow{\pi} \mathcal{PQ}_5(V) \xrightarrow{\mathrm{sym}} S^2V^* \otimes V^* \xrightarrow{p} \wedge^2 E \otimes EH$$

of an element $\alpha \otimes \beta \wedge \gamma$, $\alpha, \beta, \gamma \in V^*$, is never zero if β and γ are linearly independent over the para-quaternions $\widetilde{\mathbb{H}}$ (which is possible as $n \geq 2$). This implies that $\mathcal{PQ}_5(V) \simeq KH$, and therefore $\mathcal{PQ}_4(V) \simeq S^3 EH$. ■

Making use of the isomorphisms seen in the previous proof we find

$$\dim \mathcal{PQ}_1(V) = \dim \mathcal{PQ}_3(V) = 4n, \qquad \dim \mathcal{PQ}_2(V) = 8n,$$

$$\dim \mathcal{PQ}_4(V) = \frac{4}{3}n(n+1)(2n+1), \qquad \dim \mathcal{PQ}_5(V) = \frac{16}{3}n(n^2-1).$$

Therefore, para-quaternion Kähler structures in the composed class $\mathcal{PQ}_1 + \mathcal{PQ}_2 + \mathcal{PQ}_3$ are sections of a vector bundle whose rank grows linearly with the dimension of the manifold. This motivates the following definition.

Definition 4.2.14 *A homogeneous para-quaternion Kähler structure is called of linear type if it belongs to the class $\mathcal{PQ}_1 + \mathcal{PQ}_2 + \mathcal{PQ}_3$.*

It is easy to see that a homogeneous para-quaternion Kähler structure of linear type takes the form

$$S_XY = g(X,Y)\xi - g(Y,\xi)X - \sum_{a=1}^{3} \epsilon_a \left(g(J_aY,\xi)J_aX - g(X,J_aY)J_a\xi\right) + \sum_{a=1}^{3} g(X,\zeta^a)J_aY, \quad (4.14)$$

for some vector fields ξ and ζ^a, $a = 1, 2, 3$. Since the underlying metric g is not definite we need the following further definition.

Definition 4.2.15 *A homogeneous para-quaternion Kähler structure of linear type defined by the vector fields ξ and ζ^a, $a = 1, 2, 3$, is called*

1. *non-degenerate if $g(\xi,\xi) = 0$,*
2. *degenerate if $g(\xi,\xi) = 0$.*

A *homogeneous para-hyper-Kähler structure* is a homogeneous structure S on a para-hyper-Kähler manifold (M, g, J_1, J_2, J_3) satisfying $\widetilde{\nabla} J_a = 0$, $a = 1, 2, 3$. This implies that S satisfies (4.12) and (4.13) with $\pi^a = 0$ for $a = 1, 2, 3$. Therefore, the decomposition of $\mathcal{PHK}(V) = V^* \otimes \mathfrak{sp}(n, \mathbb{R})$ can be read in terms of the para-quaternion Kähler case. In fact,

$$\mathcal{PHK}(V) = \mathcal{PHK}_1(V) + \mathcal{PHK}_2(V) + \mathcal{PHK}_3(V),$$

where $\mathcal{PHK}_1(V)$, $\mathcal{PHK}_2(V)$, $\mathcal{PHK}_3(V)$ have the same expression as $\mathcal{PQ}_3(V)$, $\mathcal{PQ}_4(V)$ and $\mathcal{PQ}_5(V)$ respectively. A homogeneous para-hyper-Kähler structure S is said of *linear type* if it belongs to the class $\mathcal{PHK}_1$. In that case, S seen as $(1, 2)$-tensor field has the form

$$S_X Y = g(X,Y)\xi - g(Y,\xi)X - \sum_{a=1}^{3} \epsilon_a \left(g(J_a Y, \xi) J_a X - g(X, J_a Y) J_a \xi\right),$$

where $\xi = \theta^\sharp$. In addition, $S \in \mathcal{PHK}_1(V)$ is said degenerate if ξ is isotropic, and non-degenerate if ξ is non-isotropic.

4.2.6 Homogeneous Sasakian and cosymplectic structures

Let (M, g, ϕ, ξ, η) be an almost contact metric manifold, we consider homogeneous structures S satisfying Ambrose-Singer-Kiričenko equations, that is

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} S = 0, \qquad \widetilde{\nabla} \phi = 0.$$

Such structures will be called *homogeneous almost contact metric structures.*

Remark 4.2.16 *The condition $\widetilde{\nabla}\phi = 0$ is equivalent to $\widetilde{\nabla}\Phi = 0$, and implies $\widetilde{\nabla}\eta = 0$ and $\widetilde{\nabla}\xi = 0$ since*

$$\widetilde{\nabla}_X \Phi(\xi, Y) + \Phi(\widetilde{\nabla}_X \xi, Y) = 0$$

Classification of homogeneous almost contact metric structures was carried out in [26] with a representation theoretical approach. We recall that classification and obtain the corresponding classes of tensors. Hereafter we suppose $\dim M \geq 5$.

We take $V = \mathbb{R}^{2n+1}$ endowed with its standard almost contact metric structure $(\langle\ ,\ \rangle, \phi, \xi, \eta)$ as the model of the tangent space $T_x M$ at a fixed point $x \in M$. We also consider the space of $(0, 3)$-tensors on V satisfying the same algebraic symmetries as a homogeneous almost constant metric structure, that is,

$$\mathcal{S}(V) = \{S \in \otimes^3 V / \, S_{XYZ} + S_{XZY} = 0\}.$$

The condition $\widetilde{\nabla}\phi = 0$ reads

$$\nabla_X \phi = [S_X, \phi], \tag{4.15}$$

where the bracket denotes the usual commutator of endomorphisms. We thus take the subspace $\mathcal{S}_+(V) \subset \mathcal{S}(V)$ consisting of tensors such that S_X commutes with ϕ, that is

$$\mathcal{S}_+(V) = \{S \in \mathcal{S}(V) / \, S_{X\phi Y \phi Z} - S_{XYZ} = 0\}.$$

A simple computation shows that the symmetric bilinear form defined on $\mathcal{S}(V)$ is non-degenerate on $\mathcal{S}_+(V)$, so that we can consider its orthogonal complement

$$\mathcal{S}_-(V) = \{S \in \mathcal{S}(V) / \, S_{X\phi Y \phi Z} + S_{XYZ} = \eta(Y) S_{X\xi Z} + \eta(Z) S_{XY\xi}\}.$$

Identifying $\mathcal{S}(V)$ with $V^* \otimes (\wedge^2 V^*) \simeq V^* \otimes \mathfrak{so}(2p+1, 2q)$ (or $\mathfrak{so}(2p, 2q+1)$ depending on the value of ε) we have that $\mathcal{S}_+(V)$ is isomorphic to $V^* \otimes \mathfrak{u}(p, q)$, where $\mathfrak{u}(p, q)$ is seen as the Lie algebra of $U(p, q) \times \{1\} \subset O(2p+1, 2q)$ (or $O(2p, 2q+1)$). Hence $\mathcal{S}_-(V)$ is identified with $V^* \otimes \mathfrak{u}(p, q)^\perp$ (note that the Killing forms of $\mathfrak{so}(2p+1, 2q)$ and $\mathfrak{so}(2p, 2q+1)$ are non-degenerate on $\mathfrak{u}(p, q)$). Therefore $\mathcal{S}_+(V)$ is the space of tensors with the same symmetries as homogeneous almost contact metric structures on a cosymplectic manifold, and $\mathcal{S}_-(V)$ is the space of tensors with the same symmetries as $\nabla\phi$. In addition, due to (4.15) we have that $\nabla\phi$ gives the component of S in $\mathcal{S}_-(V)$. It is obvious that $\mathcal{S}_+(V)$ is invariant under the induced $U(p,q)\times\{1\}$ representation on $\mathcal{S}(V)$, hence so is $\mathcal{S}_-(V)$.

Following [26], and using the same notation as in Subsection 4.2.2, we have for $n \geq 3$ the decompositions into irreducible $U(p,q)\times\{1\}$-modules

$$\mathcal{S}_+(V) = \mathbb{R} + 2[\![\Lambda^{1,0}]\!] + [\Lambda_0^{1,1}] + [\![\Lambda_0^{2,1}]\!] + [\![B]\!],$$

$$\mathcal{S}_-(V) = 2\mathbb{R} + 2[\![\Lambda^{1,0}]\!] + 2[\Lambda_0^{1,1}] + 2[\![\Lambda^{2,0}]\!] + [\![\Lambda_0^{2,1}]\!] + [\![\Lambda^{3,0}]\!] + [\![\sigma^{2,0}]\!] + [\![A]\!].$$

For $n = 2$ these decompositions are valid except that the modules $[\![\Lambda_0^{2,1}]\!]$ and $[\![\Lambda^{3,0}]\!]$ are absent in $\mathcal{S}_-(V)$, and $[\![\Lambda_0^{2,1}]\!]$ is absent in $\mathcal{S}_+(V)$. As we indicated in Section 1.2.3, a decomposition of $\mathcal{S}_-(V)$ using real tensors is obtained in [22]. The irreducible submodule corresponding to α-Sasakian structures is the one dimensional space

$$\mathcal{C}_6(V) = \{S \in \mathcal{S}_-(V) / \, S_{XYZ} = \alpha\varepsilon(\eta(Z)\langle X,Y\rangle - \eta(Y)\langle X,Z\rangle),\ \alpha \in \mathbb{R}\}.$$

Therefore for homogeneous almost contact metric structure S on a Sasakian manifold (i.e., a 1-Sasakian manifold) the component in $\mathcal{S}_-(V)$ is given by

$$(S_-)_{XYZ} = \varepsilon(\eta(Z)\langle X,Y\rangle - \eta(Y)\langle X,Z\rangle).$$

We finally obtain a decomposition of $\mathcal{S}_+(V)$ into irreducible $U(p,q)\times\{1\}$-modules using real tensors.

Proposition 4.2.17 *For $n \geq 3$, the space $\mathcal{S}_+(V)$ decomposes into irreducible and mutually orthogonal $U(p,q)\times\{1\}$-modules as*

$$\mathcal{S}_+(V) = \mathcal{CS}_1(V) + \mathcal{CS}_2(V) + \mathcal{CS}_3(V) + \mathcal{CS}_4(V) + \mathcal{CS}_5(V) + \mathcal{CS}_6(V),$$

where

$$\begin{aligned}
\mathcal{CS}_1(V) &= \Big\{S \in \mathcal{D}_2 \,/\, S_{XYZ} = \frac{1}{2}\big(S_{ZXY} + S_{YZX} + S_{\phi ZX\phi Y} + S_{\phi Y\phi ZX}\big), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad c_{12}(S) = 0\Big\}, \\
\mathcal{CS}_2(V) &= \Big\{S \in \mathcal{D}_2 \,/\, S_{XYZ} = \langle X,Y\rangle\psi_1(Z) - \langle X,Z\rangle\psi_1(Y) + \langle X,\phi Y\rangle\psi_1(\phi Z) \\
&\qquad\qquad - \langle X,\phi Z\rangle\psi_1(\phi Y) - 2\langle\phi Y,Z\rangle\psi_1(\phi X),\ \psi_1 \in \bar{V}^*\Big\}, \\
\mathcal{CS}_3(V) &= \Big\{S \in \mathcal{D}_2 \,/\, S_{XYZ} = -\frac{1}{2}\big(S_{ZXY} + S_{YZX} + S_{\phi ZX\phi Y} + S_{\phi Y\phi ZX}\big), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad c_{12}(S) = 0\Big\}, \\
\mathcal{CS}_4(V) &= \Big\{S \in \mathcal{D}_2 \,/\, S_{XYZ} = \langle X,Y\rangle\psi_2(Z) - \langle X,Z\rangle\psi_2(Y) + \langle X,\phi Y\rangle\psi_2(\phi Z) \\
&\qquad\qquad - \langle X,\phi Z\rangle\psi_2(\phi Y) + 2\langle\phi Y,Z\rangle\psi_2(\phi X),\ \psi_2 \in \bar{V}^*\Big\}, \\
\mathcal{CS}_5(V) &= \Big\{S \in \mathcal{S}_+(V) \,/\, S_{XYZ} = \eta(X)\omega_0(Y,Z)\Big\}, \\
\mathcal{CS}_6(V) &= \Big\{S \in \mathcal{S}_+(V) \,/\, \mathrm{trace}(S_\xi) = 0\Big\}.
\end{aligned}$$

If n=2 then $\mathcal{S}_+(V) = \mathcal{CS}_2(V) + \mathcal{CS}_3(V) + \mathcal{CS}_4(V) + \mathcal{CS}_5(V) + \mathcal{CS}_6(V)$.

Proof. We first decompose $V^* = \mathbb{R}\eta + \bar{V}^*$, where $\bar{V}$ is the orthogonal complement to ξ. This gives the following orthogonal decomposition into $U(p,q)$-modules

$$V^* \otimes \mathfrak{u}(p,q) = (\mathbb{R}\eta + \bar{V}^*) \otimes \mathfrak{u}(p,q) = \mathbb{R}\eta \otimes \mathfrak{u}(p,q) + \bar{V}^* \otimes \mathfrak{u}(p,q).$$

The first summand is isomorphic to $\mathfrak{u}(p,q)$ and is identified with

$$\mathcal{D}_1 = \{S \in \mathcal{S}_+(V) / \, S_{XYZ} = \eta(X) S_{\xi YZ}\}.$$

We can thus further decompose $\mathfrak{u}(p,q) = \mathbb{R}\omega_0 + \mathfrak{su}(p,q)$ where ω_0 is the symplectic form on $\bar{V}$ inherited from Φ. This translates into $\mathcal{D}_1 = \mathcal{CS}_5 \oplus \mathcal{CS}_6$. The second summand consists of basic tensors and is identified with

$$\mathcal{D}_2 \{S \in \mathcal{S}_+(V) / \, S_{\xi YZ} = 0\}.$$

Note that $\mathcal{D}_2 \simeq \bar{V}^* \otimes \mathfrak{u}(p,q)$ can be seen as the space of tensors with the same symmetries as homogeneous pseudo-Kähler structures on $\bar{V}$, so that following subsection 4.2.2 we can decompose $\mathcal{D}_2 = \mathcal{CS}_1 + \mathcal{CS}_2 + \mathcal{CS}_3 + \mathcal{CS}_4$. ■

It is easy to see that we have isomorphisms

$$\mathcal{CS}_1(V) \simeq [\![\Lambda_0^{2,1}]\!], \quad \mathcal{CS}_2(V) \simeq \mathcal{CS}_4(V) \simeq [\![\Lambda^{1,0}]\!], \quad \mathcal{CS}_3(V) \simeq [\![B]\!],$$
$$\mathcal{CS}_5(V) \simeq \mathbb{R}, \quad \mathcal{CS}_6(V) \simeq [\Lambda_0^{1,1}],$$

so that the previous are irreducible $U(p,q) \times \{1\}$-modules.

Let (M, g, ϕ, ξ, η) be a cosymplectic manifold, and let S be a homogeneous almost contact metric structure on M, which we will simply call a *homogeneous cosymplectic structure*. Since $\nabla\phi = 0$, the S_- part of S vanishes, so that $S \in \mathcal{S}_+$. We consider the class $\mathcal{CS}_2 + \mathcal{CS}_4 + \mathcal{CS}_5$. Although this class has dimension $4n+1$ which does not grow linearly with the dimension of the manifold, it depends on two basic 1-forms and a one dimensional vertical term corresponding to the $\mathcal{CS}_5$ part. Therefore, in analogy with the pseudo-Kähler case, we call S a *homogeneous cosymplectic structure of linear type*. The corresponding $(1,2)$-vector field takes the form

$$\begin{aligned} S_X Y &= g(X,Y)\chi - g(\chi,Y)X - g(X,\phi Y)\phi\chi + g(\chi, J\phi Y)\phi X \\ &- 2g(\zeta, \phi X)\phi Y - \alpha\eta(X)\phi Y, \end{aligned} \tag{4.16}$$

for some vector fields χ and ζ, and $\alpha \in \mathbb{R}$.

On the other hand, let (M, g, ϕ, ξ, η) be a Sasakian manifold, and let S be a homogeneous almost contact metric structure on M, which we will simply call a *homogeneous Sasakian structure*. Since $\nabla_X \phi = [S_X, \phi]$ the S_- part of S belongs to $\mathcal{C}_6$ with $\alpha = 1$. We consider a homogeneous Sasakian structure whose $\mathcal{S}_+$ part belongs to the class $\mathcal{CS}_2 + \mathcal{CS}_4 + \mathcal{CS}_5$. For the same reason as before we will call S a *homogeneous Sasakian structure of linear type*. The corresponding $(1,2)$-tensor field takes the form

$$\begin{aligned} S_X Y &= g(X,Y)\chi - g(\chi,Y)X - g(X,\phi Y)\phi\chi + g(\chi, J\phi Y)\phi X \\ &- 2g(\zeta, \phi X)\phi Y - \alpha\eta(X)\phi Y + g(X,Y)\xi - \varepsilon\eta(Y)X, \end{aligned} \tag{4.17}$$

for some vector fields χ and ζ, and $\alpha \in \mathbb{R}$.

Since the metric g restricted to the contact distribution $D = \mathrm{Span}\{\xi\}^\perp$ may have signature, we have to distinguish between the following cases.

Definition 4.2.18 *A homogeneous cosymplectic (Sasakian) structure of linear type is called*

1. *non-degenerate if* $g(\chi,\chi) \neq 0$, *and*
2. *degenerate if* $g(\chi,\chi) = 0$.

Remark 4.2.19 *The same study can be done replacing the Sasakian condition by the* α*-Sasakian condition, which implies that the intrinsic torsion belongs to the class* $\mathcal{C}_6$ *with* α *not necessarily equal to* 1.

Chapter 5

Homogeneous ϵ-Kähler structures of linear type

In this chapter we study homogeneous structures of linear type on pseudo-Kähler and para-Kähler manifolds. On the one hand, we obtain that non-degenerate homogeneous pseudo-Kähler and para-Kähler structures of linear type characterize spaces of constant holomorphic and para-holomorphic sectional curvature. Moreover, if the metric is not definite, we show that the corresponding complex and para-complex space forms locally admit this kind of structures, but unlike in the Riemannian setting, the global existence is faced with the completeness of the metric. On the other hand, we completely determine the holonomy and the local form of pseudo-Kähler and para-Kähler admitting degenerate homogeneous structures of linear type. In addition we exhibit the relation between their underlying geometry and the geometry of homogeneous plane waves.

Since many features in the geometry of pseudo-Kähler and para-Kähler manifolds are very similar, it is very convenient to develop the arguments and the results simultaneously. For this reason we unify this geometries through the notion of ϵ-Kähler manifolds.

Definition 5.0.20 *Let (M, g) be a pseudo-Riemannian manifold.*

1. *An almost ϵ-Hermitian structure on (M, g) is a smooth section J of $\mathfrak{so}(TM)$ such that $J^2 = \epsilon \mathrm{Id}$.*
2. *(M, g) is called ϵ-Kähler if it admits a parallel almost ϵ-Hermitian structure J with respect to the Levi-Civita connection.*

This way, one recovers the corresponding formula or result in the pseudo-Kähler and the para-Kähler cases by substituting $\epsilon = -1$ and $\epsilon = 1$ respectively. In particular we can write a homogeneous ϵ-Kähler structure of linear type as

$$S_X Y = g(X, Y)\xi - g(\xi, Y)X + \epsilon g(X, JY)J\xi - \epsilon g(\xi, JY)JX - 2g(\zeta, JX)JY, \qquad (5.1)$$

for some vector fields ξ and ζ. The notions of degenerate and non-degenerate structures remain the same (Definitions 4.2.9 and 4.2.6). We shall also use the terms ϵ-complex and ϵ-holomorphic which include the complex and para-complex cases in the obvious way. In addition $\mathbb{C}^\epsilon$ will denote the complex and the para-complex numbers for $\epsilon = \pm 1$ respectively, where i_ϵ will stand for the corresponding imaginary unit.

5.1 The non-degenerate case

Lemma 5.1.1 *Let (M, g, J) be a connected ϵ-Kähler manifold of dimension $2n \geqslant 4$ admitting a non-degenerate homogeneous ϵ-Kähler structure of linear type. Then (M, g, J) is Einstein.*

Proof. The following proof has been adapted to the pseudo-Riemannian setting from one appearing in [29] in the Riemannian case. Equation $\widetilde{\nabla}R = 0$ reads

$$(\nabla_X R)_{YZWU} = -R_{S_XYZWU} - R_{YS_XZWU} - R_{YZS_XWU} - R_{YZWS_XU}, \tag{5.2}$$

so applying the second Bianchi identity and substituting (5.1) we have

$$\begin{aligned}0 = \underset{XYZ}{\mathfrak{S}} \big\{&2g(X,\xi)R_{YZWU} + g(X,W)R_{YZ\xi U} + g(X,U)R_{YZW\xi}\\ &+ 2\epsilon g(X,JY)R_{J\xi ZWU} + \epsilon g(X,JW)R_{YZJ\xi U} + \epsilon g(X,JU)R_{YZWJ\xi}\big\}.\end{aligned}$$

Since $g(\xi,\xi) \neq 0$ we can choose an orthonormal basis including $\xi/\sqrt{|g(\xi,\xi)|}$. Contracting the previous formula with respect to X and W and applying first Bianchi identity we obtain

$$\begin{aligned}(2n+2)R_{ZY\xi U} = &-2g(Y,\xi)Ric(Z,U) + 2g(Z,\xi)Ric(Y,U)\\ &- 2\epsilon g(Y,JZ)Ric(J\xi,U) - g(Y,U)Ric(Z,\xi)\\ &- \epsilon g(Y,JU)Ric(Z,J\xi) + g(Z,U)Ric(Y,\xi)\\ &+ \epsilon g(Z,JU)Ric(Y,J\xi),\end{aligned} \tag{5.3}$$

where Ric is the Ricci curvature. Denoting the scalar curvature by s, we can deduce $Ric(Z,\xi) = (\mathrm{s}/2n)g(Z,\xi)$ by contracting (5.3) with respect to Y and U with the same orthonormal basis as before. Setting $a = 1/(2n+2)$ and $\nu = \mathrm{s}/2n$, we can write

$$\frac{1}{a}R_{\xi U} = 2\theta \wedge Ric(U) - 2\nu\epsilon\theta(JU)\omega + bU^\flat \wedge \theta + \nu(JU)^\flat \wedge (\theta \circ J), \tag{5.4}$$

where ω is the symplectic form associated to g and J. Using the identity $R_{WUJ\xi\cdot} = R_{\xi JWU\cdot} - R_{\xi JUW\cdot}$ we can write (5.3) as

$$\begin{aligned}0 = {}&2\theta \wedge R_{WU} + W^\flat \wedge R_{\xi U} - U^\flat \wedge R_{\xi W}\\ &- 2\epsilon\omega \wedge (R_{\xi JUW} - R_{\xi JWU})\\ &- \epsilon(JW)^\flat \wedge R_{\xi JU} + \epsilon(JU)^\flat \wedge R_{\xi JW}.\end{aligned} \tag{5.5}$$

Denoting the right hand side of (5.4) by $\Xi(U)$ and substituting in (5.5) we obtain

$$\begin{aligned}0 = {}&\frac{2}{a}\theta \wedge R_{WU} + W^\flat \wedge \Xi(U) - U^\flat \wedge \Xi(W)\\ &- 2\epsilon\omega \wedge (i_W\Xi(JU) - i_U\Xi(JW))\\ &- \epsilon(JW)^\flat \wedge \Xi(JU) + \epsilon(JU)^\flat \wedge \Xi(JW).\end{aligned}$$

Taking $W = \xi$ the previous formula transforms into

$$0 = \epsilon(2g(\xi,\xi)\omega + \theta \wedge (\theta \circ J)) \wedge (Ric(JU) - \nu(JU)^\flat),$$

and contracting first with ξ and then with $J\xi$ we obtain

$$g(\xi,\xi)(Ric(JU) - \nu(JU)^\flat) = 0.$$

Since $g(\xi,\xi) \neq 0$ we deduce that the manifold is Einstein. ∎

Theorem 5.1.2 *Let (M,g,J) be a connected ϵ-Kähler manifold of dimension $2n \geqslant 4$ admitting a non-degenerate homogeneous ϵ-Kähler structure S of linear type. Then (M,g,J) has constant ϵ-holomorphic sectional curvature $c = -4g(\xi,\xi)$ and $\zeta = 0$.*

Proof. Since by the previous Lemma (M, g, J) is Einstein, formula (5.3) transforms into

$$R_{YZ\xi W} = cR^0_{YZ\xi W},$$

where $c = \mathrm{s}/(4n(n+1))$ and R^0 is the curvature of a manifold with constant ϵ-holomorphic sectional curvature equal to 4, i.e.,

$$\begin{aligned} R^0_{XYZW} = {} & g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + \epsilon g(X,JZ)g(Y,JW) \\ & - \epsilon g(x,JW)g(Y,JZ) + 2\epsilon g(X,JY)g(Z,JW). \end{aligned}$$

This implies that

$$R_{XJX}\xi = c\left\{-2g(JX,\xi)X + 2g(X,\xi)JX - 2g(X,X)J\xi\right\}. \tag{5.6}$$

On the other hand, $\widetilde{\nabla}\xi = 0$ is equivalent to $\nabla_X\xi = S_X\xi$. Using this in

$$R_{XJX}\xi = \nabla_{[X,JX]}\xi - \nabla_X\nabla_{JX}\xi + \nabla_{JX}\nabla_X\xi,$$

we get

$$R_{XJX}\xi = -g(\xi,\xi)R^0_{XJX}\xi + \Theta^\zeta_{XJX}\xi, \tag{5.7}$$

where

$$\begin{aligned} \Theta^\zeta_{XY}\xi = {} & 2g(X,J\zeta)\left\{g(Y,J\xi)\xi + g(\xi,\xi)JY + 2\epsilon g(\zeta,Y)J\xi\right\} \\ & - 2g(Y,J\zeta)\left\{g(X,J\xi)\xi + g(\xi,\xi)JX + 2\epsilon g(X,\zeta)J\xi\right\} \\ & + 2\left\{g(Y,\zeta)g(\xi,JX) - g(X,\zeta)g(\xi,JY) + 2g(X,JY)g(\xi,\zeta)\right\}J\xi. \end{aligned}$$

Taking $Y = X$ and $X \in \mathrm{Span}\{\zeta, J\zeta\}^\perp$, and comparing formulas (5.6) and (5.7), we have that $c = -g(\xi,\xi)$ and $g(\xi,\zeta) = 0$. In addition, this implies that $\Theta^\zeta_{XJX}\xi = 0$, whence $2\epsilon g(\xi,\xi)g(X,\zeta) = 0$. This together with $g(\xi,\zeta) = 0$ gives $\zeta = 0$. Let now $A = R + g(\xi,\xi)R^0$. A direct computation from (5.2) gives

$$\begin{aligned} (\nabla_X R)_{YZWU} = {} & g(Y,\xi)A_{XZWU} + g(Z,\xi)A_{YXWU} + g(W,\xi)A_{YZXU} \\ & + g(U,\xi)A_{YZWX} - g(JY,\xi)A_{JXZWU} - g(JZ,\xi)A_{YJXWU} \\ & - g(JW,\xi)A_{YZJXU} - g(JU,\xi)A_{YZWJX}. \end{aligned}$$

Since A satisfies first Bianchi identity, taking cyclic sum in X, Y, Z we obtain

$$0 = -2 \underset{XYZ}{\mathfrak{S}} g(X,\xi)A_{YZWU},$$

which is equivalent to $\theta \wedge A_{WU} = 0$. Contracting with ξ and taking into account that $A_{YZ\xi W} = 0$ we have that

$$0 = g(\xi,\xi)A_{WU},$$

hence $A_{WU} = 0$. This proves that (M, g, J) has constant ϵ-holomorphic sectional curvature $-4g(\xi,\xi)$. ∎

Remark 5.1.3 *For $\epsilon = -1$, if $g(\xi,\xi) > 0$ then $c = -4g(\xi,\xi) < 0$, so that spaces with negative definite metric and constant negative holomorphic sectional curvature cannot admit non-degenerate homogeneous pseudo-Kähler structures of linear type. Similarly, if $g(\xi,\xi) < 0$ then $c > 0$, so that spaces with positive definite metric and constant positive holomorphic sectional curvature are also excluded.*

5.2 The degenerate case

Equation $\widetilde{\nabla}R = 0$ reads

$$(\nabla_X R)_{YZWU} = -R_{S_XYZWU} - R_{YS_XZWU} - R_{YZS_XWU} - R_{YZWS_XU}, \tag{5.8}$$

so applying second Bianchi identity and substituting (5.1) we have

$$\begin{aligned} 0 = \underset{XYZ}{\mathfrak{S}} \{ & 2g(X,\xi)R_{YZWU} + g(X,W)R_{YZ\xi U} + g(X,U)R_{YZW\xi} \\ & +2\epsilon g(X,JY)R_{J\xi ZWU} + \epsilon g(X,JW)R_{YZJ\xi U} + \epsilon g(X,JU)R_{YZWJ\xi}\}. \end{aligned} \tag{5.9}$$

Since $g(\xi,\xi) = 0$, there exists an orthonormal basis $\{e_k\}$ such that $g(e_1,e_1) = 1$, $g(e_2,e_2) = -1$, and $\xi = g(\xi,e_1)(e_1+e_2)$. Whence, contracting the previous formula with respect to X and W and applying first Bianchi identity, we obtain

$$\begin{aligned} (2n+2)R_{ZY\xi U} &= -2g(Y,\xi)Ric(Z,U) + 2g(Z,\xi)Ric(Y,U) \\ &\quad -2\epsilon g(Y,JZ)Ric(J\xi,U) - g(Y,U)Ric(Z,\xi) \\ &\quad -\epsilon g(Y,JU)Ric(Z,J\xi) + g(Z,U)Ric(Y,\xi) \\ &\quad +\epsilon g(Z,JU)Ric(Y,J\xi). \end{aligned} \tag{5.10}$$

With the same orthonormal basis, contracting the previous expression with respect to Y and U we arrive to $Ric(Z,\xi) = (\mathrm{s}/2n)g(Z,\xi)$. Setting $a = 1/(2n+2)$ and $\nu = \mathrm{s}/2n$, we can write

$$\frac{1}{a}R_{\xi U} = 2\theta\wedge Ric(U) - 2\nu\epsilon\theta(JU)\omega + \nu U^\flat\wedge\theta - \epsilon\nu(JU)^\flat\wedge(\theta\circ J), \tag{5.11}$$

where ω denotes the symplectic form associated to (g,J). From first Bianchi identity we have $R_{WUJ\xi\cdot} = R_{\xi JWU\cdot} - R_{\xi JUW\cdot}$ so we can write (5.10) as

$$\begin{aligned} 0 &= 2\theta\wedge R_{WU} + W^\flat\wedge R_{\xi U} - U^\flat\wedge R_{\xi W} \\ &\quad -2\epsilon\omega\wedge(R_{\xi JUW} - R_{\xi JWU}) \\ &\quad -\epsilon(JW)^\flat\wedge R_{\xi JU} + \epsilon(JU)^\flat\wedge R_{\xi JW}. \end{aligned} \tag{5.12}$$

Denoting by $\Xi(U)$ the right hand side of (5.11) and substituting in (5.12) we obtain

$$\begin{aligned} 0 &= \frac{2}{a}\theta\wedge R_{WU} + W^\flat\wedge\Xi(U) - U^\flat\wedge\Xi(W) \\ &\quad -2\epsilon\omega\wedge(i_W\Xi(JU) - i_U\Xi(JW)) \\ &\quad -\epsilon(JW)^\flat\wedge\Xi(JU) + \epsilon(JU)^\flat\wedge\Xi(JW). \end{aligned}$$

Then, taking $W = \xi$ in the previous formula

$$0 = \epsilon(\theta\wedge(\theta\circ J))\wedge(Ric(JU) - \nu(JU)^\flat).$$

Now, since U is arbitrary, denoting $\alpha = Ric - \nu g$, one has

$$\theta\wedge(\theta\circ J)\wedge\alpha(X) = 0,$$

for any vector field X. This implies that

$$\alpha = \lambda\theta + \mu\theta\circ J,$$

for some 1-forms λ and μ. Note that since (M,g,J) is ϵ-Kähler, $\alpha = Ric - \nu g$ is symmetric and of type $(1,1)$. Imposing this to the right hand side of the previous equation we have that

$$\lambda = f\theta, \qquad \mu = -\epsilon f(\theta\circ J),$$

for some function f, so that we obtain

$$Ric = \nu g + f\left(\theta \otimes \theta - \epsilon(\theta \circ J) \otimes (\theta \circ J)\right). \tag{5.13}$$

Substituting (5.13) in (5.11)

$$\frac{1}{a} R_{\xi U} = \nu R^0_{\xi U} + P_{\xi U}, \tag{5.14}$$

where again

$$\begin{aligned} R^0_{XYZW} = {} & g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + \epsilon g(X,JZ)g(Y,JW) \\ & - \epsilon g(x,JW)g(Y,JZ) + 2\epsilon g(X,JY)g(Z,JW). \end{aligned}$$

and

$$P_{\xi U} = -2\epsilon f \theta(JU)\theta \wedge (\theta \circ J).$$

On the other hand, from $\nabla\xi = S \cdot \xi$ and (5.1), formula

$$R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

gives

$$R_{XY}\xi = -g(\xi,\xi)R^0_{XY}\xi + \Theta^\zeta_{XY}\xi = \Theta^\zeta_{XY}\xi, \tag{5.15}$$

where

$$\begin{aligned} \Theta^\zeta_{XY}\xi = & -2g(\zeta,JY)g(X,J\xi)\xi + 2g(\zeta,Y)g(X,J\xi)J\xi - 4g(\zeta,JY)g(X,\xi)J\xi \\ & + 2g(\zeta,JX)g(Y,J\xi)\xi - 2g(\zeta,X)g(Y,J\xi)J\xi + 4g(\zeta,JX)g(Y,\xi)J\xi \\ & + 4g(\xi,\zeta)g(Y,JX)J\xi - 4\epsilon g(\zeta,JY)g(\xi,X)J\xi + 4\epsilon g(\zeta,JX)g(\zeta,Y)J\xi. \end{aligned}$$

Taking $Y = JX$ and comparing (5.14) and (5.15) one finds that

$$2abg(\xi,JX) = 0, \qquad 2abg(\xi,X) = 0,$$

for every X, so that $\nu = 0$. Hence the scalar curvature vanishes. We now choose a basis

$$\{\xi, J\xi, q_1, Jq_1, X_i, JX_i\}$$

of T_pM for every $p \in M$, where $g(\xi,q_1) = 1$, $g(q_1,q_1) \neq 0$, and $\{X_i, JX_i\}$ is an orthonormal basis of $\mathrm{Span}\{\xi, J\xi, q_1, Jq_1\}^\perp$. Comparing again (5.14) and (5.15) for $X = \xi$ and $Y = Jq_1$, and for $X = J\xi$ and $Y = Jq_1$ we obtain that $g(\zeta,J\xi) = 0$ and $g(\zeta,\xi) = 0$, so that $\zeta \in \mathrm{Span}\{\xi, J\xi\}^\perp$. Taking $X = X_i$ and $Y = Jq_1$, and $X = JX_i$ and $Y = Jq_1$ we also have $g(\zeta,JX_i) = 0$ and $g(\zeta,X_i) = 0$ respectively, so that $\zeta \in \mathrm{Span}\{\xi, J\xi\}$. Finally, writing $\zeta = \lambda\xi + \mu J\xi$ for some functions λ and μ, and taking $X = q_1$ and $Y = Jq_1$ one finds $g(\zeta,Jq_1) = 0$ and $2af = -2\epsilon\lambda - 4\lambda^2$, so that

$$\zeta = \lambda\xi, \qquad f = -\frac{1}{a}\lambda(\epsilon + 2\lambda).$$

Note that equations $\widetilde{\nabla}\xi = 0$ and $\widetilde{\nabla}\zeta = 0$ imply that λ must be constant. This agrees with the fact that the Ricci form

$$\rho = f\theta \wedge (\theta \circ J)$$

is closed as (M,g,J) is ϵ-Kähler. We have proved

Proposition 5.2.1 *Let (M,g,J) be an ϵ-Kähler manifold admitting a degenerate ϵ-Kähler homogeneous structure of linear type given by* (5.1)*. Then $\zeta = \lambda\xi$ for some $\lambda \in \mathbb{R}$ and the Ricci curvature is*

$$Ric = -\frac{1}{a}\lambda(\epsilon + 2\lambda)\left(\theta \otimes \theta - \epsilon(\theta \circ J) \otimes (\theta \circ J)\right),$$

where $a = 1/(\mathrm{dim}M + 2)$ and $\theta = \xi^\flat$. In particular the scalar curvature vanishes.

Since $\nu = 0$, formula (5.14) becomes

$$R_{ZY\xi U} = aP_{ZY\xi U} = -2a\epsilon f(\theta \wedge (\theta \circ J) \otimes (\theta \circ J))(Z, Y, U).$$

Looking again to formula (5.9) we obtain

$$\begin{aligned} -\mathfrak{S}_{XYZ} 2g(X,\xi)R_{YZWU} = \mathfrak{S}_{XYZ} 2af \Big\{ &(\theta \wedge (\theta \circ J)) \otimes (X^\flat \wedge (\theta \circ J))(Y, Z, W, U) \\ &+ \epsilon(\theta \wedge (\theta \circ J)) \otimes (JX^\flat \wedge (\theta))(Y, Z, W, U) \\ &- 2\epsilon g(X, JY)\theta \otimes (\theta \wedge (\theta \circ J))(Z, W, U) \} . \end{aligned} \tag{5.16}$$

Substituting this in (5.8) and after a quite long computation, which can be found in the Appendix at the end of this manuscript,

$$\nabla_X R = 4\theta(X) \otimes (R - \frac{1}{2}ag \boxtimes Ric) - 2a\epsilon \Big((X^\flat \wedge (\theta \circ J)) \odot \rho + (JX^\flat \wedge (\theta)) \odot \rho \Big), \tag{5.17}$$

where ρ is the Ricci form and $\boxtimes$ stands for the ϵ-complex Kulkarni-Nomizu product defined as

$$\begin{aligned} h \boxtimes k(X_1, X_2, X_3, X_4) = {} & h(X_1, X_3)k(X_2, X_4) + h(X_2, X_4)k(X_1, X_3) \\ & - h(X_1, X_4)k(X_2, X_3) - h(X_2, X_3)k(X_1, X_4) \\ & - \epsilon h(X_1, JX_3)k(X_2, JX_4) - \epsilon h(X_2, JX_4)k(X_1, JX_3) \\ & + \epsilon h(X_1, JX_4)k(X_2, JX_3) + \epsilon h(X_2, JX_3)k(X_1, JX_4) \\ & - 2\epsilon h(X_1, JX_2)k(X_3, JX_4) - 2\epsilon h(X_3, JX_4)k(X_1, JX_2), \end{aligned}$$

for h and k symmetric $(0, 2)$-tensors.

With the help of (5.17) we now compute some terms of the curvature tensor of g. We again choose a basis

$$\{\xi, J\xi, q_1, Jq_1, X_i, JX_i\}$$

of T_pM for every $p \in M$. Taking the symmetric sum with respect to X, Y, Z in (5.17) we have

$$\begin{aligned} 0 = {} & 4\theta(X) \left(R_{YZWU} - 2ag \boxtimes Ric_{YZWU}\right) \\ & - 2a\epsilon \Big((X^\flat \wedge (\theta \circ J)) \odot \rho + (JX^\flat \wedge (\theta)) \odot \rho \Big) (Y, Z, W, U) \\ & - 2a\epsilon \Big((Y^\flat \wedge (\theta \circ J)) \odot \rho + (JY^\flat \wedge (\theta)) \odot \rho \Big) (Z, X, W, U) \\ & - 2a\epsilon \Big((Z^\flat \wedge (\theta \circ J)) \odot \rho + (JZ^\flat \wedge (\theta)) \odot \rho \Big) (X, Y, W, U). \end{aligned}$$

Setting $Y, Z \in \mathrm{Span}\,\{\xi, J\xi\}^\perp$ we obtain

$$R_{YZWU} = -8a\epsilon g(Y, JZ)\rho(W, U), \qquad Y, Z \in \mathrm{Span}\{\xi, J\xi\}^\perp \tag{5.18}$$

for every W, U. On the other hand setting $X = q_1$, $Y = Jq_1$ and $Z \in \mathrm{Span}\, X_i, JX_i$ we find

$$\begin{aligned} R_{YZWU} = {} & af\left(g(Z, W)\theta(JU) - g(Z, U)\theta(JW) - g(Z, JW)\theta(U)\right. \\ & \left. + g(Z, JU)\theta(W)\right), \end{aligned}$$

for every W, U, so that

$$\begin{aligned} R_{q_1ZWU} = {} & af\left(g(JZ, U)\theta(JW) - g(JZ, W)\theta(JU) + \epsilon g(Z, U)\theta(W)\right. \\ & \left. - \epsilon g(Z, W)\theta(U)\right) \end{aligned} \tag{5.19}$$

for $Z \in \mathrm{Span}\{X_i, JX_i\}$ and all W, U.

Proposition 5.2.2 *(M,g,J) is Ricci-flat.*

Proof. Let $g(q_1,q_1)=b$ and suppose for the sake of simplicity that $b>0$ (the case $b<0$ is analogous). Denoting $q_2=Jq_1$, we choose an orthonormal basis

$$\left\{\sqrt{b}\left(\xi-\frac{q_1}{b}\right),\sqrt{b}\left(J\xi-\frac{q_2}{b}\right),\frac{q_1}{\sqrt{b}},\frac{q_2}{\sqrt{b}},X_i,JX_i\right\}$$

of T_pM for every $p\in M$, which has signature $(-1,\epsilon,1,-\epsilon,\varepsilon^i,-\epsilon\varepsilon^i)$ where $g(X_i,X_i)=\varepsilon^i\in\{\pm1\}$. We compute the Ricci curvature by contracting the curvature tensor with respect to this orthonormal basis and using (5.18) and (5.19):

$$\begin{aligned}
Ric(W,U) &= -R\left(W,\sqrt{b}\left(\xi-\frac{q_1}{b}\right),U,\sqrt{b}\left(\xi-\frac{q_1}{b}\right)\right)\\
&\quad+\epsilon R\left(W,\sqrt{b}\left(\xi-\frac{q_2}{b}\right),U,\sqrt{b}\left(\xi-\frac{q_2}{b}\right)\right)\\
&\quad+R\left(W,\frac{q_1}{\sqrt{b}},U,\frac{q_1}{\sqrt{b}}\right)-\epsilon R\left(W,\frac{q_2}{\sqrt{b}},U,\frac{q_2}{\sqrt{b}}\right)\\
&\quad+\varepsilon^i R(W,X_i,U,X_i)-\epsilon\varepsilon^i R(W,JX_i,U,JX_i)\\
&= 4af\left(\theta\otimes\theta-\epsilon(\theta\circ J)\otimes(\theta\circ J)\right)\\
&\quad+2af\epsilon\sum_i\varepsilon^i\left(\theta\otimes\theta-\epsilon(\theta\circ J)\otimes(\theta\circ J)\right)\\
&= (4a+2a\epsilon\sum_i\varepsilon^i)Ric(W,U).
\end{aligned}$$

We deduce that if $Ric(W,U)\neq 0$ then $4a+2a\epsilon\sum_i\varepsilon^i=1$, therefore

$$\dim M+2=4+2a\epsilon\sum_i\varepsilon^i,$$

whence

$$\dim M=2+2a\epsilon\sum_i\varepsilon^i<\dim M.$$

Since this is impossible we conclude that $Ric=0$. ■

Corollary 5.2.3 *The only possible values for λ are $\lambda=0$ and $\lambda=-\frac{\epsilon}{2}$.*

In the next section we shall study the cases $\lambda=0$ and $\lambda=-\frac{\epsilon}{2}$ separately.

Proposition 5.2.4 *The curvature tensor of g is given by*

$$R=k(\theta\wedge(\theta\circ J))\otimes(\theta\wedge(\theta\circ J)),$$

for some function k. Moreover, if $k\neq 0$ the holonomy algebra of g is given by

$$\mathfrak{hol}\cong\mathbb{R}\begin{pmatrix} i_\epsilon & i_\epsilon & 0\\ -i_\epsilon & -i_\epsilon & 0\\ 0 & 0 & 0_n\end{pmatrix},$$

which is a one dimensional subalgebra of $\mathfrak{su}(1,1)\subset\mathfrak{su}(p,q)$, $p+q=n+2$, for $\epsilon=-1$, and $\mathfrak{sl}(2,\mathbb{R})\subset\mathfrak{sl}(n+2,\mathbb{R})$ for $\epsilon=1$.

Proof. Since (M,g,J) is Ricci-flat (5.17) becomes

$$\nabla R=4\theta\otimes R.$$

Taking symmetric sum in the previous formula and applying second Bianchi identity we have that $\theta \wedge R_{WU} = 0$ for every W, U. But from the ϵ-Kähler symmetries of R we also have $(\theta \circ J) \wedge R_{WU} = 0$. These force the curvature to be of the form

$$R = k(\theta \wedge (\theta \circ J)) \otimes (\theta \wedge (\theta \circ J)),$$

for some function k.

On the other hand, since (M, g, J) is real analytic, the infinitesimal holonomy algebra coincides with the holonomy algebra (see [38, Ch. II]). Recall that the infinitesimal holonomy algebra at $p \in M$ is defined as $\mathfrak{hol}' = \bigcup_{l=0}^{\infty} \mathfrak{m}_l$, where

$$\mathfrak{m}_0 = \mathrm{Span}\{R_{XY} / \, X, Y \in T_pM\}$$

and

$$\mathfrak{m}_l = \mathrm{Span}\left\{\mathfrak{m}_{l-1} \cup \{(\nabla_{Z_l} \dots \nabla_{Z_1} R)_{XY} / \, Z_1, \dots, Z_l, X, Y \in T_pM\}\right\}.$$

As a simple computation shows one has

$$\nabla\theta = \theta \otimes \theta + (2\lambda + \epsilon)(\theta \circ J) \otimes (\theta \circ J).$$

It is easy to see that this together with the recurrent formula $\nabla R = 4\theta \otimes R$ implies that $\mathfrak{m}_0 = m_1 = \dots = \mathfrak{m}_l$ for every $l \in \mathbb{N}$, so that $\mathfrak{hol}' = \mathfrak{m}_0$. Now, since $R = k(\theta \wedge (\theta \circ J)) \otimes (\theta \wedge (\theta \circ J))$ the space $\mathfrak{m}_0$ is the one dimensional space generated by the endomorphism

$$\begin{array}{rrcl} A: & T_pM & \to & T_pM \\ & \xi, J\xi & \mapsto & 0 \\ & q_1 & \mapsto & J\xi \\ & q_2 & \mapsto & \epsilon\xi \\ & X_i, JX_i & \mapsto & 0. \end{array}$$

This endomorphism is expressed as

$$\frac{1}{b}\begin{pmatrix} i_\epsilon & i_\epsilon & 0 \\ -i_\epsilon & -i_\epsilon & 0 \\ 0 & 0 & 0_n \end{pmatrix}$$

with respect to the ϵ-complex orthonormal basis

$$\left\{\frac{1}{\sqrt{|b|}}(q_1 + \epsilon i_\epsilon q_2), \left(\frac{1}{\sqrt{|b|}} q_1 - s\sqrt{|b|}\xi\right) + \epsilon i_\epsilon \left(\frac{1}{\sqrt{|b|}} q_2 - s\sqrt{|b|} J\xi\right), X_i + \epsilon i_\epsilon J X_i\right\},$$

where $g(q_1, q_1) = b$ and s is the sign of b. ■

As a consequence of Proposition 5.2.4 we have that for $\epsilon = \pm 1$ and $\lambda = 0, -\frac{\epsilon}{2}$ (M, g, J) is an *Osserman manifold* with a 2-step nilpotent Jacobi operator. It is also easy to see that (M, g, J) is VSI (vanishing scalar invariants). Finally, it is worth noting that making use of Theorem 1.2.14, if (M, g, J) is connected and simply-connected, then it is the product of a $2n$-dimensional ϵ-complex flat and totally geodesic manifold and a 4-dimensional Walker ϵ-Kähler manifold with a parallel null ϵ-complex vector field. The similarities between this kind of manifolds and the structure of plane waves will be explored in detail in Section 5.4.

5.2.1 Local form of the metrics

We have seen (Propositions 5.2.1 and 5.2.2) that an ϵ-Kähler manifold (M, g, J) admitting a degenerate homogeneous ϵ-Kähler structure of linear type is Ricci-flat and satisfies $\zeta = \lambda\xi$ for some constant $\lambda \in \mathbb{R}$. As stated in Corollary 5.2.3 this implies that the only possible values for λ are $\lambda = 0$ and $\lambda = -\frac{\epsilon}{2}$. Hereafter M is supposed to be non-flat and of dimension $2n + 4$.

The case $\lambda = -\frac{\epsilon}{2}$:

Substituting the value $\lambda = -\frac{\epsilon}{2}$ in (5.1) we have

$$S_X Y = g(X,Y)\xi - g(\xi,Y)X + \epsilon g(X,JY)J\xi - \epsilon g(\xi,JY)JX + \epsilon g(\xi,JX)JY.$$

Condition $\widetilde{\nabla}\xi = 0$ then implies

$$\nabla\xi = \theta\otimes\xi,$$

which gives

$$\nabla\theta = \theta\otimes\theta,$$
$$\nabla(\theta\circ J) = \theta\otimes(\theta\circ J).$$

In particular $d\theta = 0$, so that fixing a point $p\in M$ there is a neighborhood $\mathcal{U}$ and a function $v:\mathcal{U}\to\mathbb{R}$ such that $\theta = dv$. We consider

$$w_1 = e^{-v},$$

whence $dw_1 = -e^{-v}dv = -w_1\theta$. We now take

$$dw_1\circ J = -w_1(\theta\circ J).$$

Differentiating we obtain

$$d(dw_1\circ J) = -dw_1\wedge(\theta\circ J) - w_1 d(\theta\circ J) = w_1\theta\wedge(\theta\circ J) - w_1\theta\wedge(\theta\circ J) = 0.$$

Therefore, reducing $\mathcal{U}$ if necessary, there is a function $w_2:\mathcal{U}\to\mathbb{R}$ such that $dw_2 = \epsilon dw_1\circ J$. We consider the function $w = w_1 + i_\epsilon w_2$. Then $dw = dw_1 + \epsilon i_\epsilon(dw_1\circ J)$, so that dw is of type $(1,0)$ with respect to J and $w:\mathcal{U}\to\mathbb{C}^\epsilon$ is ϵ-holomorphic. In addition

$$\nabla dw = -dw_1\otimes\theta - w_1\nabla\theta - i_\epsilon dw_1\otimes(\theta\circ J) - i_\epsilon w_1\nabla(\theta\circ J) = 0,$$

i.e., dw is a nowhere vanishing parallel 1-form.

The function $w:\mathcal{U}\to\mathbb{C}^\epsilon$ defines a foliation of $\mathcal{U}$ by ϵ-complex hypersurfaces $\mathcal{H}_\tau = w^{-1}(\tau)$, $\tau\in\mathbb{C}^\epsilon$ (for those τ with $w^{-1}(\tau)$ non empty). Note that since the tangent space to $\mathcal{H}_\tau$ is given by the kernel of dw, the hypersurfaces $\mathcal{H}_\tau$ are tangent to the distribution $\mathrm{Span}\{\xi, J\xi\}^\perp$. We consider the vector field

$$Z = \mathrm{grad}(w_1) = dw_1^\sharp.$$

It is easy to see that by construction

$$JZ = -\epsilon\,\mathrm{grad}(w_2).$$

These vector fields are written as

$$Z = -w_1\xi,$$
$$JZ = -w_1 J\xi,$$

so that

$$\nabla Z = -dw_1\otimes\xi - w_1\nabla\xi = w_1\theta\otimes\xi - w_1\theta\otimes\xi = 0,$$

and thus also $\nabla JZ = 0$. This implies in particular that Z and JZ are commuting ϵ-holomorphic Killing vector fields.

We now look at the holonomy of g at p, which was computed in Proposition 5.2.4. Using the same notation as before we denote $E = \mathrm{Span}\{\xi, J\xi, q_1, q_2\}\subset T_pM$. This subspace is invariant under the holonomy action and so is $E^\perp$. In fact, the holonomy action on $E^{\perp}$ is trivial. This implies that, using the parallel transport with respect to ∇, we can extend an orthonormal basis $\{(X_a)_{|p}, (JX_a)_{|p},\ a = 1,\ldots,n\}$ of $E^{\perp}$ to an orthonormal reference $\{X_a, JX_a,\ a = 1,\ldots,n\}$ on $\mathcal{U}$ such that $\nabla X_a = 0 = \nabla JX_a$, $a = 1,\ldots,n$. In particular they are commuting ϵ-holomorphic Killing vector fields. In addition, let γ be any smooth curve on $\mathcal{U}$, we have

$$\left.\frac{d}{dt}\right|_{t=0} \left(dw(X_a)_{\gamma(t)}\right) = \left(\nabla_{\dot\gamma(t)} dw\right)(X_a) + dw\left(\nabla_{\dot\gamma(t)} X_a\right) = 0,$$

whence the functions $dw(X_a)$ are constant along γ and take the value 0 at p. This implies that X_a and thus JX_a are tangent to the foliation $\mathcal{H}_{\tau}$. Finally note that since they are parallel, X_a and JX_a commute with Z and JZ.

We have thus constructed a set of commuting para-holomorphic Killing vector fields $\{Z, JZ, X_a, JX_a\}$ tangent to $\mathcal{H}_{\tau}$. Therefore, reducing $\mathcal{U}$ if necessary, we can take ϵ-complex coordinates $\{w, z, z^a\}$ on U such that $\partial_z = \frac{1}{2}(Z + \epsilon i_{\epsilon} JZ)$, $\partial_{z^a} = \frac{1}{2}(X_a + \epsilon i_{\epsilon} JX_a)$. Note that since the distributions $\mathrm{Span}\{\partial_w, \partial_z\}$ and $\mathrm{Span}\{\partial_{z^a}, a = 1,\ldots,n\}$ are invariant by holonomy, the vector fields X_a and JX_a are orthogonal to $\mathrm{Span}\{\partial_w, \partial_z\}$. We write $z = z^1 + i_{\epsilon} z^2$, $z^a = x^a + i_{\epsilon} y^a$ and $w = w^1 + i_{\epsilon} w^2$, and rearrange the coordinates as $\{z^1, z^2, w^1, w^2, x^a, y^a\}$. The metric with respect to these coordinates is

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -\epsilon & 0 & \ldots & 0 \\ 1 & 0 & b & 0 & 0 & \ldots & 0 \\ 0 & -\epsilon & 0 & -\epsilon b & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & & \Sigma & \\ 0 & 0 & 0 & 0 & & & \end{pmatrix}, \tag{5.20}$$

for some function b, where

$$\Sigma = \mathrm{diag}(\begin{pmatrix} \varepsilon^a & 0 \\ 0 & -\epsilon\varepsilon^a \end{pmatrix}, a = 1,\ldots,n),$$

with $\varepsilon^a = g(X_a, X_a) \in \{\pm 1\}$. In addition, the ϵ-complex structure reads

$$J = \begin{pmatrix} 0 & \epsilon & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \epsilon \\ & & & 1 & 0 \end{pmatrix}. \tag{5.21}$$

Imposing that ∂_{z^1}, ∂_{z^2}, ∂_{x^a} and ∂_y^a are parallel, it is easy to see that b does not depend on z^1, z^2, x^a, y^a.

Finally, computing the curvature tensor with respect to those coordinates we obtain

$$R = \frac{1}{2}\Delta^{\epsilon} b (dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2),$$

where

$$\Delta^{\epsilon} = -\epsilon \frac{\partial^2}{\partial (w^1)^2} + \frac{\partial^2}{\partial (w^2)^2}.$$

Denoting $F = \Delta^{\epsilon} b$ and taking into account that dw^1 and dw^2 are parallel, we have that

$$\nabla R = \frac{1}{2} dF \otimes (dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2).$$

Recall that formula (5.17) together with the Ricci-flatness of g gave that

$$\nabla R = 4\theta \otimes R.$$

Comparing these two formulas for ∇R we have that

$$dF = 4F\theta,$$

where θ can be written as

$$\theta = -\frac{1}{w^1} dw^1.$$

Note that by construction $w^1 \neq 0$. The system of partial differential equations is thus

$$\begin{aligned} \frac{\partial F}{\partial w^1} &= -\frac{4}{w_1} F \\ \frac{\partial F}{\partial w^2} &= 0, \end{aligned}$$

which has solution

$$F = \frac{R_0}{(w^1)^4},$$

for some constant $R_0 \in \mathbb{R}$. We have thus proved

Proposition 5.2.5 *Let (M, g, J) be an ϵ-Kähler manifold of dimension $2n+4$, $n \geq 0$, admitting a degenerate homogeneous ϵ-Kähler structure of linear type with $\zeta = -\frac{\epsilon}{2}\xi$. Then each $p \in M$ has a neighborhood ϵ-holomorphically isometric to an open subset of $(\mathbb{C}^\epsilon)^{n+2}$ with the ϵ-Kähler metric*

$$g = dw^1 dz^1 - \epsilon dw^2 dz^2 + b(dw^1 dw^1 - \epsilon dw^2 dw^2) + \sum_{a=1}^{n} \varepsilon^a (dx^a dx^a - \epsilon dy^a dy^a), \quad (5.22)$$

where $\varepsilon^a = \pm 1$, and the function b only depends on the coordinates $\{w^1, w^2\}$ and satisfies

$$\Delta^\epsilon b = \frac{R_0}{(w^1)^4}$$

for $R_0 \in \mathbb{R} - \{0\}$.

The strongly degenerate case $\lambda = 0$:

Substituting the value $\lambda = 0$ in (5.1) we have that the homogeneous structure S takes the form

$$S_X Y = g(X, Y)\xi - g(\xi, Y)X + \epsilon g(X, JY)J\xi - \epsilon g(\xi, JY)JX.$$

Condition $\widetilde{\nabla}\xi = 0$ then implies

$$\nabla \xi = \theta \otimes \xi,$$

which gives

$$\begin{aligned} \nabla\theta &= \theta \otimes \theta - (\theta \circ J) \otimes (\theta \circ J), \\ \nabla(\theta \circ J) &= \theta \otimes (\theta \circ J) - \epsilon(\theta \circ J) \otimes \theta. \end{aligned}$$

We consider the ϵ-complex form $\alpha = \theta + \epsilon i_\epsilon(\theta \circ J)$, which is of type $(1,0)$ with respect to the ϵ-complex structure J. As a straightforward computation shows, $\nabla\alpha = \alpha \otimes \alpha$ so that $d\alpha = 0$. This implies in particular that α is an ϵ-holomorphic 1-form. Fixing a point $p \in M$, by the closeness of α, there is a neighborhood $\mathcal{U}$ of p and an ϵ-holomorphic function $v : \mathcal{U} \to \mathbb{C}^\epsilon$ such that $\alpha = dv$. We consider the ϵ-holomorphic function

$$w = e^{-v},$$

where the the exponential must read $e^{x+i_\epsilon y} = e^x(\cos y + i_\epsilon \sin y)$ for $\epsilon = -1$ and $e^{x+i_\epsilon y} = e^x(\cosh y + i_\epsilon \sinh y)$ for $\epsilon = 1$. Differentiating we obtain that

$$dw = -w\alpha,$$

so that $\nabla dw = 0$. This means that dw is a nowhere vanishing parallel ϵ-holomorphic 1-form on $\mathcal{U}$. The function $w : \mathcal{U} \to \mathbb{C}^\epsilon$ defines a foliation of $\mathcal{U}$ by ϵ-complex hypersurfaces $\mathcal{H}_\tau = w^{-1}(\tau)$, $\tau \in \mathbb{C}^\epsilon$ (for those τ with $w^{-1}(\tau)$ non empty). Note that since the tangent space to $\mathcal{H}_\tau$ is given by the kernel of dw, the hypersurfaces $\mathcal{H}_\tau$ are tangent to the distribution $\mathrm{Span}\{\xi, J\xi\}^\perp$. Writing $w = w^1 + i_\epsilon w^2$, we take the vector fields

$$Z = (dw^1)^\sharp, \qquad JZ = -\epsilon(dw^2)^\sharp.$$

These vector fields are obviously tangent to the foliation given by $\mathcal{H}_\tau$, and since $\nabla dw = 0$ we have $\nabla Z = 0$ and $\nabla JZ = 0$. This implies in particular that Z, JZ are commuting ϵ-holomorphic Killing vector fields.

Making use of Proposition 5.2.4, and by the same arguments used for the case $\lambda = -\epsilon/2$, we take coordinates $\{w, z, z^a\}$, $a = 1, \ldots, n$, such that $\partial_{z^a} = \frac{1}{2}(Z + \epsilon i_\epsilon JZ)$ and $\nabla \partial_{z^a} = 0$. Writing $z = z^1 + i_\epsilon z^2$, $z^a = x^a + i_\epsilon y^a$, and $w = w^1 + i_\epsilon w^2$, with respect to real coordinates $\{z^1, z^2, w^1, w^2, x^a, y^a\}$ the metric g and the complex structure J take the form (5.20) and (5.21) respectively, where b does not depend on z^1, z^2, x^a, y^a. As a straightforward computation shows,

$$R = \frac{1}{2}\Delta^\epsilon b(dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2),$$

and

$$\theta = \frac{-1}{(w^1)^2 - \epsilon(w^2)^2}(w^1 dw^1 - \epsilon w_2 dw^2).$$

Finally, imposing $\nabla R = 4\theta \otimes R$ and denoting $F = \Delta^\epsilon b$, we obtain the system of partial differential equations

$$\frac{\partial F}{\partial w^1} = \frac{-4w^1}{(w^1)^2 - \epsilon(w^2)^2} F$$

$$\frac{\partial F}{\partial w^2} = \frac{4\epsilon w^2}{(w^1)^2 - \epsilon(w^2)^2} F,$$

which has solution

$$F = \frac{R_0}{((w^1)^2 - \epsilon(w^2)^2)^2},$$

for some constant $R_0 \in \mathbb{R}$. Note that since $w = e^{-v}$ we always have $(w^1)^2 - \epsilon(w^2)^2 \neq 0$. We have thus proved

Proposition 5.2.6 *Let (M, g, J) be an ϵ-Kähler manifold of dimension $2n+4$, $n \geq 0$, admitting a strongly degenerate homogeneous ϵ-Kähler structure of linear type S. Then each $p \in M$ has a neighborhood ϵ-holomorphically isometric to an open subset of $(\mathbb{C}^\epsilon)^{n+2}$ with the ϵ-Kähler metric*

$$g = dw^1 dz^1 - \epsilon dw^2 dz^2 + b(dw^1 dw^1 - \epsilon dw^2 dw^2) + \sum_{a=1}^{n} \varepsilon^a (dx^a dx^a - \epsilon dy^a dy^a), \quad (5.23)$$

where $\varepsilon^a = \pm 1$, and the function b only depends on the coordinates $\{w^1, w^2\}$ and satisfies

$$\Delta^\epsilon b = \frac{R_0}{((w^1)^2 - \epsilon(w^2)^2)^2}$$

for $R_0 \in \mathbb{R} - \{0\}$.

The manifold $((\mathbb{C}^\epsilon)^{n+2}, g)$

Propositions 5.2.6 and 5.2.5 give the local forms (5.23) and (5.22) of the metric of a manifold with a degenerate homogeneous ϵ-Kähler structure of linear type. This motivates the study of the space $(\mathbb{C}^\epsilon)^{2+n}$ endowed with this particular ϵ-Kähler metric, which can thus be understood as the simplest instance of this type of manifolds. In particular, the goal is to study the singular nature of this spaces. We shall restricts ourselves to the Lorentz ϵ-Kähler case, i.e., metrics of index 2. Throughout this section $\|w\|_\lambda$ must be understood as

$$\|w\|^2_\lambda = \begin{cases} w_1^2 - \epsilon w_2^2 & \text{for} \quad \lambda = 0, \\ w_1^2 & \text{for} \quad \lambda = -\epsilon/2. \end{cases} \tag{5.24}$$

In addition Δ^ϵ shall stand for the differential operator

$$\Delta^\epsilon = -\epsilon \frac{\partial^2}{\partial w_1^2} + \frac{\partial^2}{\partial w_2^2}.$$

We thus consider the manifold $(\mathbb{C}^\epsilon)^{2+n} = (\mathbb{R}^{2n+4}, J_0)$, where J_0 is the standard ϵ-complex structure, with real coordinates $\{z^1, z^2, w^1, w^2, x^a, y^a\}$, endowed with the metric

$$g = dw^1 dz^1 - \epsilon dw^2 dz^2 + b(dw^1 dw^1 - \epsilon dw^2 dw^2) + \sum_{a=1}^{n} (dx^a dx^a - \epsilon dy^a dy^a), \tag{5.25}$$

where b is a function of the variables (w^1, w^2) satisfying

$$\Delta^\epsilon b = \frac{R_0}{\|w\|^4_\lambda}, \qquad R_0 \in \mathbb{R} - \{0\}. \tag{5.26}$$

As computed before, the curvature $(1,3)$-tensor field of g is

$$R = \frac{1}{2} \frac{R_0}{\|w\|^4_\lambda} \left((dw^1 \wedge dw^2) \otimes (dw^1 \otimes \partial_{z^2}) + \epsilon (dw^1 \wedge dw^2) \otimes (dw^2 \otimes \partial_{z^1}) \right).$$

As $R_0 \neq 0$, the curvature exhibits a singular behavior at

$$\mathcal{S} = \{\|w\|_\lambda = 0\}.$$

This set can be understood as a singularity of g in the cosmological sense: the geodesic deviation equation is governed by the components $R^{z^j}_{w^1 w^2 w^i}$, $i, j = 1, 2$, of the curvature tensor field, making the tidal forces infinite at $\mathcal{S}$. Indeed, we can compute a component of the curvature tensor with respect to an orthonormal parallel frame along a geodesic reaching the singular set in finite time, and see that it is singular (see [55]). Let γ be the geodesic with initial value $\gamma(0) = (0, 0, 1, 0, \ldots, 0)$ and $\dot\gamma = (0, 0, -1, 0, \ldots, 0)$. It is easy to see that this geodesic is of the form

$$\gamma(t) = (z^1(t), z^2(t), 1 - t, 0, x^a(t), y^a(t))$$

for some functions $z^1(t), z^2(t), x^a(t), y^a(t)$, $a = 1, \ldots, n$, and reaches the singular set $\mathcal{S}$ at $t = 1$. Let

$$E(t) = W^1(t)\partial_{w^1} + W^2(t)\partial_{w^2} + Z^1(t)\partial_{z^1} + Z^2(t)\partial_{z^2} + X^a(t)\partial_{x^a} + Y^a(t)\partial_{y^a}$$

be a vector field along γ. E is parallel if the following equations hold:

$$\begin{aligned} 0 &= \dot W^1, & 0 &= \dot W^2, \\ 0 &= \dot Z^1 - W^1 \Gamma^{z^1}_{w^1 w^1} - W^2 \Gamma^{z^1}_{w^1 w^2}, & 0 &= \dot Z^2 - W^1 \Gamma^{z^2}_{w^1 w^1} - W^2 \Gamma^{z^2}_{w^1 w^2}, \\ 0 &= \dot X^a, & 0 &= \dot Y^a. \end{aligned}$$

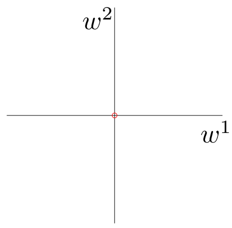
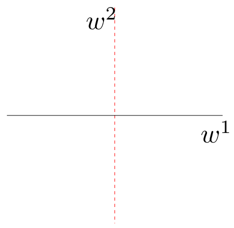
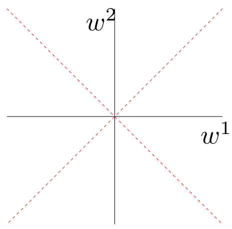
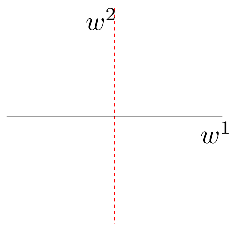
Singular set $\mathcal{S}$	$\lambda = 0$	$\lambda = -\frac{\epsilon}{2}$
$\epsilon = -1$	 $\mathcal{S} : (w^1)^2 + (w^2)^2 = 0$	 $\mathcal{S} : w^1 = 0$
$\epsilon = 1$	 $\mathcal{S} : (w^1)^2 - (w^2)^2 = 0$	 $\mathcal{S} : w^1 = 0$

Table 5.1: Singular sets

We can thus obtain an orthonormal parallel frame $\{E_1(t), \ldots, E_{4+2n}(t)\}$ with $E_1(t)$ and $E_2(t)$ of the form

$$\begin{aligned} E_1(t) =& \frac{1}{\sqrt{|b(0)|}}\partial_{w^1} + Z_1^1(t)\partial_{z^1} + Z_1^2(t)\partial_{z^2} + X_1^a\partial_{x^a} + Y_1^a\partial_{y^a}, \\ E_2(t) =& \frac{1}{\sqrt{|b(0)|}}\partial_{w^2} + Z_2^1(t)\partial_{z^1} + Z_2^2(t)\partial_{z^2} + X_2^a\partial_{x^a} + Y_2^a\partial_{y^a}, \end{aligned}$$

where $E_1(0) = \frac{1}{\sqrt{|b(0)|}}\partial_{w^1}$, $E_2(0) = \frac{1}{\sqrt{|b(0)|}}\partial_{w^2}$, and $b(0)$ is the value of b at $w = 0$. The curvature tensor applied to $E_1(t), E_2(t)$ is

$$R_{E_1(t)E_2(t)E_1(t)E_2(t)} = \frac{R_0}{2b(0)^2}\frac{1}{||w(t)||_\lambda^4} = \frac{R_0}{2b(0)^2}\frac{1}{(1-t)^4},$$

which is singular at $t = 1$.

Note that $(\mathbb{C}^\epsilon)^{2+n} - \mathcal{S}$ is connected and not simply-connected for $\epsilon = -1$ and $\lambda = 0$ while it is not connected nor simply-connected for the other values. Moreover, $(\mathbb{C}^\epsilon)^{2+n} - \mathcal{S}$ has two connected components for $\lambda = -\epsilon/2$ and $\epsilon = \pm 1$ and four connected components for $\lambda = 0$ and $\epsilon = 1$.

We finally show that degenerate homogeneous ϵ-Kähler structures of linear type are realized on the ϵ-Kähler manifold $((\mathbb{C}^\epsilon)^{2+n} - \mathcal{S}, g)$.

Proposition 5.2.7 *For every data (b, R_0) satisfying (5.26), $((\mathbb{C}^\epsilon)^{2+n} - \mathcal{S}, g)$ admits a degenerate ϵ-Kähler homogeneous structure of linear type.*

Proof. Let

$$\xi = \begin{cases} \frac{-1}{(w^1)^2 - \epsilon(w^2)^2}(w^1\partial_{z^1} + w^2\partial_{z^2}) & \lambda = 0, \\ -\frac{1}{w^1}\partial_{z^1} & \lambda = -\frac{\epsilon}{2}. \end{cases}$$

We take the tensor field

$$S_X Y = g(X,Y)\xi - g(\xi,Y)X + \epsilon g(X,JY)J\xi - \epsilon g(\xi,JY)JX - 2\lambda g(\xi,JX)JY.$$

A straightforward computation shows that $\widetilde{\nabla}\xi = 0$ and $\widetilde{\nabla}R = 0$, where $\widetilde{\nabla} = \nabla - S$, so that S is a degenerate homogeneous ϵ-Kähler structure of linear type (see (4.3) and (4.5)). ■

5.3 Infinitesimal models, homogeneous models and completeness

Recall the definition of infinitesimal model, transvection algebra and homogeneous model associated to a homogeneous structure (Section 2.3). The aim of this section is to prove the following results.

Theorem 5.3.1 *With the exception of $\mathbb{C}\mathrm{P}^n_0$ and $\mathbb{C}H^n_n$, the indefinite ϵ-complex space forms $\mathbb{C}P^n_p$, $\mathbb{C}H^n_p$ and $\widetilde{\mathbb{C}}P^n$ locally admit a non-degenerate homogeneous ϵ-Kähler structure of linear type.*

Theorem 5.3.2 *Let (M,g,J) be a connected and simply-connected ϵ-Kähler manifold with $\dim M \geqslant 4$ admitting a non-degenerate homogeneous ϵ-Kähler structure of linear type. If g is not definite then (M,g,J) is not complete. On the other hand, the homogeneous model associated to a degenerate homogeneous ϵ-Kähler structure of linear type is not complete.*

Remark 5.3.3 *Note that Theorem 5.3.2 implies that the indefinite ϵ-complex space forms $\mathbb{C}P^n_p$, $\mathbb{C}H^n_p$ and $\widetilde{\mathbb{C}}P^n$ do not admit a globally defined homogeneous ϵ-Kähler structure of linear type except for the definite cases $\mathbb{C}P^n_n$ and $\mathbb{C}H^n_0$.*

We now explain the general procedure to prove Theorems 5.3.1 and 5.3.2. This procedure will be then specified for each case: degenerate and non-degenerate, and pseudo-Kähler and para-Kähler. The same procedure will be used for the case of pseudo-quaternion and para-quaternion structures later in Section 6.2.

Procedure for the proof of Theorems 5.3.1 and 5.3.2.

The first step is to explicitly compute the infinitesimal model and the transvection algebra $\mathfrak{g} = T_pM \oplus \mathfrak{hol}^{\widetilde{\nabla}}$ associated to a homogeneous ϵ-Kähler structure of linear type. This is done by obtaining the expression for $\widetilde{R} = R - R^S$. Denoting $\mathfrak{h} = \mathfrak{hol}^{\widetilde{\nabla}}$, we next show that the transvection algebra $(\mathfrak{g},\mathfrak{h})$ is regular, that is, H is a closed subgroup of G, where G is the simply-connected Lie group with Lie algebra $\mathfrak{g}$ and H is its connected Lie subgroup with Lie algebra $\mathfrak{h}$. In order to prove that we obtain a matrix realization of $\mathfrak{h}$ and $\mathfrak{g}$ in $\mathfrak{gl}(N,\mathbb{R})$ for some $N \in \mathbb{N}$, and we exponentiate it to see that the connected Lie subgroup of $GL(N,\mathbb{R})$ with Lie algebra $\mathfrak{h}$ is closed in $GL(N,\mathbb{R})$, whence H must be closed in G. For the degenerate case this is done using the adjoint representation. For the non-degenerate case, by Remark 1.2.18, we will only need to consider the spaces $\mathbb{C}\mathrm{H}^n_p$ and $\widetilde{\mathbb{C}}\mathrm{P}^n$, since by Theorem 5.1.2 our spaces of linear type are locally $\pm$-isometric to one of these models. Recalling the expressions (1.1) and (1.2) of $\mathbb{C}\mathrm{H}^n_s$ and $\widetilde{\mathbb{C}}\mathrm{P}^n$ as symmetric spaces $Isom/Isot$, we identify $\mathfrak{g}$ with a subalgebra of $\mathfrak{isom}$, in such a way that $\mathfrak{h}$ is the intersection of $\mathfrak{g}$ and $\mathfrak{isot}$. This gives a matrix realization of $\mathfrak{g}$ and $\mathfrak{h}$, and subgroups $G \subset Isom$ and $H \subset Isot$, and we find that H is closed in $Isom$ (which is closed in $Gl(m,\mathbb{R})$). We can thus take the homogeneous model G/H associated to S. Continuing with the non-degenerate case, the orbit of $p = eIsot$ in the model space $Isom/Isot$ is just G/H. Counting dimensions one sees that G/H is an open subset of

Isom/*Isot*. Since by construction G/H admits a non-degenerate homogeneous ϵ-Kähler structure of linear type, this would prove Theorem 5.3.1.

We now return to the general case. (M, g) is locally isometric to the homogeneous space G/H (see Proposition 3.1.12) and when (M, g) is simply-connected and complete, so it will be globally isometric to G/H. To prove Theorem 5.3.2 we show that G/H is not complete. We consider a Lie algebra involution $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ with $\sigma(\mathfrak{h}) \subset \mathfrak{h}$ and restricting to an isometry for the $\mathrm{Ad}(H)$-invariant metric on T_pM. The map σ determines a Lie group involution $\sigma\colon G \to G$ with $\sigma(H) \subset H$, and an involution σ on the homogeneous space G/H. Denote the fixed-point set of σ on X by X^σ. Then the homogeneous spaces G^σ/H^σ and $(G/H)^\sigma$ are isometric. However, σ is an isometry, so $(G/H)^\sigma$ is a totally geodesic submanifold of G/H. By considering a sequence of such Lie algebra involutions, we can construct a chain of totally geodesic submanifolds

$$\cdots \subset ((G/H)^{\sigma_1})^{\sigma_2} \subset (G/H)^{\sigma_1} \subset G/H.$$

In our cases, we use this technique to construct a totally geodesic submanifold that we can show is not complete (Lemmas 5.3.4 and 5.3.5). It follows that G/H is not geodesically complete. ■

Lemma 5.3.4 *The Lie group K with Lie algebra $\mathfrak{k} = \mathrm{Span}\{A, V\}$, $[A, V] = V$, and left invariant metric given by*

$$g(A, A) = 1, \qquad g(V, V) = -1, \qquad g(A, V) = 0,$$

is not geodesically complete, time-like complete, null complete nor space-like complete.

Proof. The Levi-Civita connection of this metric is

$$\nabla_A A = 0, \qquad \nabla_A V = 0, \qquad \nabla_V A = -V, \qquad \nabla_V V = -A.$$

Let γ be a curve in K and $\dot\gamma$ its derivative. We write $\dot\gamma(t) = \gamma_1(t)A + \gamma_2(t)V$. The geodesic equation thus implies

$$\begin{cases} \dot\gamma_1 - \gamma_2^2 = 0 \\ \dot\gamma_2 - \gamma_1\gamma_2 = 0. \end{cases}$$

The solution to this system with space-like initial value $\gamma_1(0) = 0$, $\gamma_2(0) = 1$ is $\gamma_1(t) = \tan(t)$, $\gamma_2 = 1/\cos(t)$ which is defined for $-\pi/2 < t < \pi/2$. On the other hand, the null initial value $\gamma_1(0) = 1 = \gamma_2(0)$, has solution $\gamma_1(t) = \gamma_2(t) = 1/(1-t)$ which is only defined for $t < 1$. Finally, the time-like initial value $\gamma_1(0) = 1$, $\gamma_2(0) = r$, $0 < r < 1$, has $x(t) = s\coth(st + k)$, $y(t) = s/\sinh(st + k)$, where $s = \sqrt{1 - r^2}$, $\tanh k = s$. These solutions are only defined for $t \neq -k/s$. ■

Lemma 5.3.5 *The Lie group K with Lie algebra $\mathfrak{k} = \mathrm{Span}\{U, V\}$, $[U, V] = \mu(V - U)$, where $\mu \in \mathbb{R}^+$, and left invariant metric given by*

$$g(U, U) = s = \pm 1, \qquad g(V, V) = -s, \qquad g(U, V) = 0,$$

is not geodesically complete, time-like complete, null complete nor space-like complete.

Proof. The Levi-Civita connection of g is

$$\begin{aligned} \nabla_U U &= -\tfrac{1}{\sqrt{|b(p)|}} V, & \nabla_U V &= -\tfrac{1}{\sqrt{|b(p)|}} U, \\ \nabla_V V &= -\tfrac{1}{\sqrt{|b(p)|}} U, & \nabla_V U &= -\tfrac{1}{\sqrt{|b(p)|}} V. \end{aligned}$$

Let γ be a curve in K and $\dot\gamma$ its tangent vector. Setting $\dot\gamma(t) = u(t)U + v(t)V$, the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$ implies

$$\begin{aligned} \dot u - \tfrac{1}{\sqrt{|b(p)|}}(uv + v^2) &= 0 \\ \dot v - \tfrac{1}{\sqrt{|b(p)|}}(uv + u^2) &= 0. \end{aligned}$$

Changing variables to $x = u + v$ and $y = v - u$ the equations transform into

$$\dot{x} - \frac{1}{\sqrt{|b(p)|}} x^2 = 0$$
$$\dot{y} + \frac{1}{\sqrt{|b(p)|}} xy = 0.$$

Space-like and time-like initial values are obtained for example for $x(0) = 1$ and $y(0) = \pm 1$, and a null initial value is obtained for example for $x(0) = 1$ and $y(0) = 0$. For each of those cases the solutions for x is

$$x = \sqrt{|b(p)|} \frac{1}{1 - \mu t},$$

for some constant $c \in \mathbb{R}$, which is only defined for $t \neq 1/\mu$. ■

We now specify all the components involved in the proof of Theorems 5.3.1 and 5.3.2 for each case. Due to differences we treat them separately. The convention

$$R_{XY}Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z,$$
$$R^S_{XY}Z = S_{S_X Y - S_Y X}Z - S_X S_Y Z + S_Y S_X Z,$$

will be used.

5.3.1 The non-degenerate para-Kähler case

During this subsection $\widetilde{\mathbb{C}}$ denotes the set of para-complex numbers, e stands for the imaginary para-complex unit, so $e^2 = +1$, and $\bar{z}$ denotes the para-complex conjugation of $z \in \widetilde{\mathbb{C}}$.

We first compute the infinitesimal model associated to S. Using formula (5.1) with $\zeta = 0$ and $\epsilon = 1$, we obtain by direct calculation

$$\begin{aligned} R^S_{XY}Z = {} & g(\xi,\xi)\left\{g(Y,Z)X - g(X,Z)Y + g(Y,JZ)JX - g(X,JZ)JY\right\} \\ & - 2g(X,JY)\left\{g(\xi,JZ)\xi + g(\xi,Z)J\xi\right\}, \end{aligned}$$

and since (M,g,J) has constant para-holomorphic sectional curvature we have

$$\widetilde{R}_{XY}Z = -2g(X,JY)\left\{g(\xi,\xi)JZ - g(\xi,JZ)\xi - g(\xi,Z)J\xi\right\}.$$

Now, $\widetilde{R}_{XY}\xi = 0$ and thus $\widetilde{R}_{XY}$ acts trivially on $\mathbb{R}^2 = \operatorname{Span}\{\xi, J\xi\}$. On the other hand for $Z \in \operatorname{Span}\{\xi, J\xi\}^\perp$, one has

$$\widetilde{R}_{XY}Z = -2g(X,JY)g(\xi,\xi)JZ,$$

so that $\widetilde{R}_{XY}$ acts on $U = \operatorname{Span}\{\xi, J\xi\}^\perp$ as $-2g(X,JY)g(\xi,\xi)J$. We conclude that $\mathfrak{hol}^{\widetilde{\nabla}}$ is one dimensional and is generated by the element $\mathcal{J} = \frac{1}{2g(\xi,\xi)^2}\widetilde{R}_{\xi J\xi}$. The remaining brackets are

$$\begin{aligned} [Z_1, Z_2] &= 2g(Z_1, JZ_2)L_0, \quad & [\xi, J\xi] &= 2g(\xi,\xi)L_0, \\ [\xi, Z] &= g(\xi,\xi)JZ & [J\xi, Z] &= g(\xi,\xi)JZ, \end{aligned} \tag{5.27}$$

where $Z_1, Z_2, Z \in U$ and $L_0 = J\xi - g(\xi,\xi)\mathcal{J}$. The transvection algebra is thus

$$\mathfrak{g} = \mathbb{R}\mathcal{J} \oplus \operatorname{Span}\{\xi, J\xi\} \oplus U.$$

On the other hand, the description (1.2) of $\widetilde{\mathbb{C}}P^n$ as a symmetric space has Cartan decomposition

$$\mathfrak{sl}(n+1,\mathbb{R}) = \mathfrak{s}(\mathfrak{gl}(n,\mathbb{R}) \oplus \mathfrak{gl}(1,\mathbb{R})) \oplus \mathfrak{m} \subset \mathfrak{so}(n+1,n+1),$$

with
$$\mathfrak{m} = \left\{ \begin{pmatrix} 0_n & v \\ -v^* & 0 \end{pmatrix} \middle| \, v \in \widetilde{\mathbb{C}}^n \right\}.$$
We write $\widetilde{\mathbb{C}}^n = \mathbb{R}^n + e\mathbb{R}^n$. The algebra $\mathfrak{sl}(n+1, \mathbb{R})$ decomposes as
$$\mathfrak{sl}(n+1, \mathbb{R}) = \mathfrak{s}\left(\mathfrak{gl}(n, \mathbb{R}) \oplus \mathfrak{gl}(1, \mathbb{R})\right) \oplus \mathfrak{a} \oplus \mathfrak{n}_1 \oplus \mathfrak{n}_2,$$
where
$$\mathfrak{a} = \mathbb{R}A_0, \qquad A_0 = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & 0 & e \\ 0 & e & 0 \end{pmatrix},$$
is a maximal $\mathbb{R}$-diagonalisable subalgebra of $\mathfrak{m}$, and
$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} 0_{n-1} & -ev & v \\ -ev^* & 0 & 0 \\ -v^* & 0 & 0 \end{pmatrix} \middle| \, v \in \widetilde{\mathbb{C}}^{n-1} \right\}, \quad \mathfrak{n}_2 = \left\{ \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & -eb & b \\ 0 & -b & eb \end{pmatrix} \middle| \, b \in \mathbb{R} \right\},$$
are the eigenspaces of the positive restricted roots $\Sigma^+ = \{\lambda, 2\lambda\}$ with $\lambda(A_0) = 1$.

We shall identify $\mathfrak{g}$ with a subalgebra of $\mathfrak{sl}(n+1, \mathbb{R})$ following arguments analogous to those in [17]. First it is obvious that $\mathcal{J} \in \mathfrak{s}\left(\mathfrak{gl}(n, \mathbb{R}) \oplus \mathfrak{gl}(1, \mathbb{R})\right)$, and since $\mathcal{J}$ acts trivially on $\mathrm{Span}\{\xi, J\xi\}$ and effectively on U, the space U can be identified with $\mathfrak{n}_1$ and $\mathrm{Span}\{\xi, J\xi\} \subset \mathbb{R}\mathcal{J} + \mathfrak{a} + \mathfrak{n}_2$. Now, from (5.27) it easily follows that $L_0 \in \mathfrak{n}_2$, and since ξ has only real eigenvalues on $\mathfrak{g}$, we can take $\xi = g(\xi, \xi)A_0$ up to a Lie algebra automorphism. Let
$$X = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & -e & 1 \\ 0 & -1 & e \end{pmatrix}.$$
Using a Lie algebra automorphism we can take $L_0 = X$ which gives $J\xi = X + g(\xi, \xi)\mathcal{J}$. Finally, identifying U with $\mathfrak{n}_1$ and $\mathfrak{n}_1$ with $\widetilde{\mathbb{C}}^{n-1}$ in the obvious way, we have from (5.27) $[v, w] = 2g(v, Jw)X$. From the matrix expression of $\mathfrak{n}_1$ we obtain $[v, w] = -2\langle v, ew\rangle X$, where $\langle v, w\rangle = \mathrm{Re}\sum_j \bar{v}_j w_j$, $v, w \in U \equiv \mathfrak{n}_1 \equiv \widetilde{\mathbb{C}}^{n-1}$. Comparing this two expressions we conclude that J is acting on U as multiplication by $-e$, therefore $\mathcal{J}$ must be
$$\mathcal{J} = \frac{e}{n+1}\,\mathrm{diag}((-2)^{n-1}, (n-1)^2),$$
with powers denoting multiplicities. Exponentiating the Lie algebra spanned by $\mathcal{J}$ we obtain a closed subgroup of $SL(n+1, \mathbb{R})$.

Regarding the Lie algebra involutions involved in the proof of Theorem 5.3.2 we take $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ given by
$$\mathcal{J} \mapsto -\mathcal{J}, \qquad A_0 \mapsto A_0, \qquad X + g(\xi, \xi)\mathcal{J} \mapsto -\left(X + g(\xi, \xi)\mathcal{J}\right),$$
$$v \mapsto -\bar{v}, \quad v \in \mathfrak{n}_1 \equiv \widetilde{\mathbb{C}}^{n-1}$$
and $\tau\colon \mathfrak{g}^\sigma \to \mathfrak{g}^\sigma$ with
$$A_0 \mapsto A_0, \qquad (v_1, \dots, v_{n-2}, v_{n-1})^T \mapsto (-v_1, \dots, -v_{n-2}, v_{n-1})^T.$$
We thus have
$$\mathfrak{k} = (\mathfrak{g}^\sigma)^\tau = \left\{ \begin{pmatrix} 0_{n-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & es \\ 0 & 0 & 0 & et \\ 0 & -es & et & 0 \end{pmatrix} \middle| \, s, t \in \mathbb{R} \right\},$$
and the chain of totally geodesic submanifolds
$$K = (G^\sigma)^\tau \subset G^\sigma = (G/H)^\sigma \subset G/H,$$
where K is as in Lemma 5.3.4, and is incomplete.

5.3.2 The non-degenerate pseudo-Kähler case

During this subsection i denotes the imaginary complex unit. The computations for the infinitesimal model are completely analogous to those in the previous subsection setting $\epsilon = -1$. We obtain that

$$\widetilde{R}_{XY}Z = 2g(X, JY)g(\xi, \xi)JZ,$$

so that $\mathfrak{hol}^{\widetilde{\nabla}}$ is the one dimensional Lie algebra generated by $\mathcal{J} = \frac{1}{2g(\xi,\xi)^2}\widetilde{R}_{\xi J\xi}$. The remaining brackets are

$$\begin{aligned} [Z_1, Z_2] &= -2g(Z_1, JZ_2)L_0, \quad & [\xi, J\xi] &= 2g(\xi, \xi)L_0, \\ [\xi, Z] &= g(\xi, \xi)JZ & [J\xi, Z] &= g(\xi, \xi)JZ, \end{aligned} \tag{5.28}$$

where $Z_1, Z_2, Z \in U$ and $L_0 = J\xi - g(\xi, \xi)\mathcal{J}$. The transvection algebra is

$$\mathfrak{g} = \mathbb{R}\mathcal{J} \oplus \mathrm{Span}\{\xi, J\xi\} \oplus U,$$

where $U = \mathrm{Span}\{\xi, J\xi\}^{\perp}$. On the other hand, recall description (1.1) of $\mathbb{C}\mathrm{H}^n_s$ as symmetric space. The Riemannian case $\mathbb{C}\mathrm{H}^n_0$ is studied in [17]. We then suppose $s > 0$, and for the sake of simplicity we also suppose $2s < n - 1$, the opposite case is analogous. Let

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \Sigma = \mathrm{diag}\big((1)^{n-2s-1}, (\varepsilon)^{s+1}\big).$$

We have

$$\mathfrak{su}(n-s, s+1) = \{C \in \mathfrak{gl}(n+1, \mathbb{C}) /\, C^*\Sigma + \Sigma C = 0,\ \mathrm{Tr}(C) = 0\},$$

so that $\mathfrak{su}(n-s, s+1)$ decomposes as

$$\mathfrak{su}(n-s, s+1) = \mathfrak{s}\left(\mathfrak{u}(n-s, s) \oplus \mathfrak{u}(1)\right) \oplus \mathfrak{a} \oplus \mathfrak{n}_1 \oplus \mathfrak{n}_2,$$

where $\mathfrak{a} = RA_0$, $A_0 = \mathrm{diag}(0, \ldots, 0, 1, -1)$,

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} 0_{n-1} & 0 & v \\ -\left(\Sigma' v\right)^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle|\, v \in \mathbb{C}^{n-1} \right\}, \quad \mathfrak{n}_2 = \left\{ \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & 0 & ib \\ 0 & 0 & 0 \end{pmatrix} \middle|\, b \in \mathbb{R} \right\},$$

for $\Sigma' = \mathrm{diag}\big((1)^{n-2s-1}, (\varepsilon)^s\big)$. As in the para-Kähler case we identify $\mathfrak{g}$ with a subalgebra of $\mathfrak{su}(n-s, s+1)$. More precisely we have that U is identified with $\mathfrak{n}_1$, $\xi = g(\xi, \xi)A_0$, and $J\xi = L_0 + g(\xi, \xi)A_0$ with

$$L_0 = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}.$$

In addition, from the matrix representation of $\mathfrak{n}_1$ we obtain

$$\mathcal{J} = \frac{i}{n+1}\,\mathrm{diag}\big((-2)^{n-1}, (n-1)^2\big),$$

so that the Lie algebra spanned by $\mathcal{J}$ gives a closed subgroup of $SU(n-s, s+1)$.

Regarding the Lie algebra involutions involved in the proof of Theorem 5.3.2 we take $\sigma \colon \mathfrak{g} \to \mathfrak{g}$ defined by

$$\mathcal{J} \mapsto -\mathcal{J}, \qquad A_0 \mapsto A_0, \qquad X + g(\xi, \xi)\mathcal{J} \mapsto -\big(X + g(\xi, \xi)\mathcal{J}\big),$$
$$v \mapsto \overline{v}, \quad v \in \mathfrak{n}_1 \equiv \mathbb{C}^{n-1}$$

and $\tau\colon \mathfrak{g}^\sigma \to \mathfrak{g}^\sigma$ with

$$A_0 \mapsto A_0, \qquad (v_1, \dots, v_{n-2}, v_{n-1})^T \mapsto (-v_1, \dots, -v_{n-1}, -v_{n-2})^T.$$

Then

$$\mathfrak{k} = (\mathfrak{g}^\sigma)^\tau = \left\{ \left. \begin{pmatrix} 0_{n-3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & -t \\ 0 & t & -t & s & 0 \\ 0 & 0 & 0 & 0 & -s \end{pmatrix} \right| s, t \in \mathbb{R} \right\},$$

and we have the following chain of totally geodesic submanifolds:

$$K = (G^\sigma)^\tau \subset G^\sigma = (G/H)^\sigma \subset G/H,$$

where K is as in Lemma 5.3.4.

5.3.3 The degenerate case with $\lambda = -\frac{\epsilon}{2}$

Denoting $p_1 = \xi$ and $p_2 = J\xi$, for the sake of simplicity we choose $x \in M$ such that with respect to the basis $\{p_1, p_2, q_1, q_2, X_a, JX_a\}$ and its dual $\{p^1, p^2, q^1, q^2, X^a, JX^a\}$ the curvature is written

$$R_x = R_0 q^1 \wedge q^2 \otimes (q^1 \otimes p_2 + \epsilon q^2 \otimes p_1).$$

Let $\mathfrak{h} = \mathfrak{hol}^{\widetilde{\nabla}}$. Substituting $\lambda = -\epsilon/2$ in (5.1) we obtain by direct calculation that the non-vanishing terms of $\widetilde{R}$ are:

$$\begin{array}{rrcl} \widetilde{R}_{p_2 q_1}: & q_1 & \mapsto & 2p_2 \\ & q_2 & \mapsto & 2\epsilon p_1 \\ & X_a & \mapsto & 0 \\ & JX_a & \mapsto & 0 \\ & p_1, p_2 & \mapsto & 0 \end{array} \qquad \begin{array}{rrcl} \widetilde{R}_{q_1 q_2}: & q_1 & \mapsto & (R_0 - b(p))p_2 \\ & q_2 & \mapsto & (R_0 - b(p))\epsilon p_1 \\ & X_a & \mapsto & -JX_a \\ & JX_a & \mapsto & -\epsilon X_a \\ & p_1, p_2 & \mapsto & 0 \end{array}$$

$$\begin{array}{rrcl} \widetilde{R}_{q_2 X_a}: & q_1 & \mapsto & -JX_a \\ & q_2 & \mapsto & -\epsilon X_a \\ & X_a & \mapsto & p_2 \\ & JX_a & \mapsto & \epsilon p_1 \\ & p_1, p_2 & \mapsto & 0 \end{array} \qquad \begin{array}{rrcl} \widetilde{R}_{q_2 JX_a}: & q_1 & \mapsto & -\epsilon X_a \\ & q_2 & \mapsto & -\epsilon JX_a \\ & X_a & \mapsto & \epsilon p_1 \\ & JX_a & \mapsto & \epsilon p_2 \\ & p_1, p_2 & \mapsto & 0 \end{array}$$

$$\begin{array}{rrcl} \widetilde{R}_{X_a JX_a}: & q_1 & \mapsto & -2p_2 \\ & q_2 & \mapsto & -2\epsilon p_1 \\ & X_a & \mapsto & 0 \\ & JX_a & \mapsto & 0 \\ & p_1, p_2 & \mapsto & 0, \end{array}$$

so that $\dim\mathfrak{h} = 2n + 2$. Choosing endomorphisms

$$A = 2(q^1 \otimes p_2 + \epsilon q^2 \otimes p_1), \quad B_a = \widetilde{R}_{q_2 X_a}, \quad C_a = \widetilde{R}_{q_2 JX_a},$$

$$K = \frac{1}{2}(R_0 - b(p))A - \sum_a (X^a \otimes JX_a + \epsilon JX^a \otimes X_a)$$

as basis of $\mathfrak{h}$, the transvection algebra $\mathfrak{g}$ has non-vanishing brackets

$$
\begin{aligned}
&[B_a, C_a] = \epsilon A, \quad [B_a, K] = -C_a, \quad [C_a, K] = -\epsilon B_a,\\
&[A, q_1] = 2p_2, \quad [A, q_2] = 2\epsilon p_1,\\
&[B_a, q_1] = -JX_a, \quad [B_a, q_2] = -\epsilon X_a, \quad [B_a, X_a] = -p_2, \quad [B_a, JX_a] = -\epsilon p_1,\\
&[C_a, q_1] = -\epsilon X_a, \quad [C_a, q_2] = -\epsilon y_a, \quad [C_a, X_a] = \epsilon p_1, \quad [C_a, JX_a] = \epsilon p_2,\\
&[K, X_a] = JX_a, \quad [K, JX_a] = \epsilon X_a,\\
&[p_1, q_1] = -p_1, \quad [p_2, q_1] = -3p_2 - A, \quad [p_2, q_2] = -2\epsilon p_1,\\
&[q_1, q_2] = 2b(p)p_2 - q_2 - \tfrac{1}{2}(R_0 - b(p))A + K,\\
&[q_1, X_a] = X_a, \quad [q_1, JX_a] = JX_a,\\
&[q_2, X_a] = 2JX_a - B_a, \quad [q_2, JX_a]\, 2\epsilon X_a - C_a,\\
&[X_a, JX_a] = 2p_2 + A.
\end{aligned}
$$

One can check that $\mathfrak{g}$ is a solvable Lie algebra with a 3-step nilradical. Since $\mathfrak{g}$ has trivial center, the adjoint representation is faithful and provides a matrix realization of $\mathfrak{g}$. With respect to this realization, a straightforward computation shows that by exponentiation of $\mathfrak{h}$ we obtain a connected Lie group H which is closed inside $GL(4n+6, \mathbb{R})$, so that $(\mathfrak{g}, \mathfrak{h})$ is regular. For instance, the matrix realization of $\mathfrak{h}$ for $n = 2$ is

$$
\begin{pmatrix}
0 & 0 & 0 & 2\lambda_A\epsilon & \lambda_{C_1}\epsilon & -\lambda_{B_1}\epsilon & \lambda_{C_2}\epsilon & -\lambda_{B_2}\epsilon & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2\lambda_A & 0 & -\lambda_{B_1} & \lambda_{C_1}\epsilon & -\lambda_{B_2} & \lambda_{C_2}\epsilon & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\lambda_{C_1}\epsilon & -\lambda_{B_1}\epsilon & 0 & \lambda_K\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\lambda_{B_1} & -\lambda_{C_1}\epsilon & \lambda_K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\lambda_{C_2}\epsilon & -\lambda_{B_2}\epsilon & 0 & 0 & 0 & \lambda_K\epsilon & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\lambda_{B_2} & -\lambda_{C_2}\epsilon & 0 & 0 & \lambda_K & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_{C_1}\epsilon & -\lambda_{C_2}\epsilon & \lambda_{B_1}\epsilon & \lambda_{B_2}\epsilon & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_K\epsilon & 0 & -\lambda_{C_1}\epsilon\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_K\epsilon & -\lambda_{C_2}\epsilon\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_K & 0 & 0 & 0 & -\lambda_{B_1}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_K & 0 & 0 & -\lambda_{B_2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

for $\lambda_A, \lambda_{B_1}, \lambda_{B_2}, \lambda_{C_1}, \lambda_{C_2}, \lambda_K \in \mathbb{R}$. Regarding the Lie algebra involutions needed for the proof of Theorem 5.3.2, we take

$$
\begin{array}{rrcl}
\sigma: & \mathfrak{g} & \to & \mathfrak{g}\\
& A & \mapsto & -A\\
& B_a & \mapsto & -B_a\\
& C_a & \mapsto & C_a\\
& K & \mapsto & -K\\
& p_1 & \mapsto & p_1\\
& p_2 & \mapsto & -p_2\\
& q_1 & \mapsto & q_1\\
& q_2 & \mapsto & -q_2\\
& X_a & \mapsto & X_a\\
& JX_a & \mapsto & -JX_a,
\end{array}
$$

and

$$
\begin{array}{rrcl}
\theta: & \mathfrak{g}^\sigma & \to & \mathfrak{g}^\sigma\\
& C_a & \mapsto & -C_a\\
& p_1 & \mapsto & p_1\\
& q_1 & \mapsto & q_1\\
& X_a & \mapsto & -X_a.
\end{array}
$$

The subalgebra of fixed points is $\mathfrak{k} = (\mathfrak{g}^\sigma)^\theta = \operatorname{span}\{p_1, q_1\}$, and we have the chain of totally geodesic submanifolds

$$
K \subset \widetilde{G^\sigma}, \qquad G^\sigma = (G/H)^\sigma \subset G/H.
$$

Let s be the sign of $b(p)$. We define the left-invariant vector fields $U = 1/(\sqrt{|b(p)|})q_1$, $V = U - s\sqrt{|b(p)|}p_1$ in $\mathfrak{k}$. We thus have $< U,U >= s$, $< V,V >= -s$, $< U,V >= 0$, and $[U,V] = \frac{1}{\sqrt{|b(p)|}}(V-U)$, where $< \, , \, >$ stands for the bilinear form inherited by $\mathfrak{k}$ from $\mathfrak{g}^\sigma$. The Lie algebra $\mathfrak{k}$ is as in Lemma 5.3.5 with $\mu = \frac{1}{\sqrt{|b(p)|}}$, so that K is not geodesically complete.

5.3.4 The degenerate case with $\lambda = 0$

Denoting again $p_1 = \xi$ and $p_2 = J\xi$, we choose $x \in M$ such that with respect to the basis $\{p_1, p_2, q_1, q_2, X_a, JX_a\}$ and its dual $\{p^1, p^2, q^1, q^2, X^a, JX^a\}$ the curvature is written

$$R_x = R_0 q^1 \wedge q^2 \otimes (q^1 \otimes p_2 + \epsilon q^2 \otimes p_1).$$

Let $\mathfrak{h} = \mathfrak{hol}^{\widetilde{\nabla}}$, substituting $\lambda = 0$ in (5.1) we obtain by direct calculation that the non-vanishing terms of $\widetilde{R}$ are:

$$\widetilde{R}_{p_1q_2}: \begin{array}{rcl} q_1 & \mapsto & -2p_2 \\ q_2 & \mapsto & -2\epsilon p_1 \\ X_a & \mapsto & 0 \\ JX_a & \mapsto & 0 \\ p_1, p_2 & \mapsto & 0 \end{array} \qquad \widetilde{R}_{p_2q_1}: \begin{array}{rcl} q_1 & \mapsto & 2p_2 \\ q_2 & \mapsto & 2\epsilon p_1 \\ X_a & \mapsto & 0 \\ JX_a & \mapsto & 0 \\ p_1, p_2 & \mapsto & 0 \end{array}$$

$$\widetilde{R}_{q_1q_2}: \begin{array}{rcl} q_1 & \mapsto & (R_0 - 2b(p))p_2 \\ q_2 & \mapsto & (R_0 - 2b(p))\epsilon p_1 \\ X_a & \mapsto & 0 \\ JX_a & \mapsto & 0 \\ p_1, p_2 & \mapsto & 0 \end{array} \qquad \widetilde{R}_{X_aJX_a}: \begin{array}{rcl} q_1 & \mapsto & -2p_2 \\ q_2 & \mapsto & -2\epsilon p_1 \\ X_a & \mapsto & 0 \\ JX_a & \mapsto & 0 \\ p_1, p_2 & \mapsto & 0, \end{array}$$

so that $\dim\mathfrak{h} = 1$. Choosing the endomorphism

$$A = 2(q^1 \otimes p_2 + \epsilon q^2 \otimes p_1)$$

as basis of $\mathfrak{h}$, the transvection algebra $\mathfrak{g}$ has non-vanishing brackets

$$\begin{aligned} &[A, q_1] = 2p_2, \quad [A, q_2] = 2\epsilon p_1, \\ &[p_1, q_1] = -p_1, \quad [p_1, q_2] = p_2 + A, \\ &[p_2, q_1] = -3p_2 - A, \quad [p_2, q_2] = -\epsilon p_1, \\ &[q_1, q_2] = 2b(p)p_2 - \tfrac{1}{2}(R_0 - 2b(p))A, \\ &[q_1, X_a] = X_a, \quad [q_1, JX_a] = JX_a, \\ &[q_2, X_a] = JX_a, \quad [q_2, JX_a] = \epsilon X_a, \\ &[X_a, JX_a] = 2p_2 + A. \end{aligned}$$

One can check that $\mathfrak{g}$ is a solvable Lie algebra with a 2-step nilradical. Since $\mathfrak{g}$ has trivial center, the adjoint representation is faithful and provides a matrix realization of $\mathfrak{g}$. The matrix realization of $\mathfrak{h}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 2\epsilon t & 0 & \dots & 0 \\ 0 & 0 & 2t & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

for $t \in \mathbb{R}$. Exponentiating we obtain a connected Lie group H which is closed in $GL(2n+5, \mathbb{R})$, so that $(\mathfrak{g}, \mathfrak{h})$ is regular. Regarding the Lie algebra involutions needed for the proof of Theorem 5.3.2, we take

$$\begin{array}{rrcl} \sigma: & \mathfrak{g} & \to & \mathfrak{g} \\ & A & \mapsto & -A \\ & p_1 & \mapsto & p_1 \\ & p_2 & \mapsto & -p_2 \\ & q_1 & \mapsto & q_1 \\ & q_2 & \mapsto & -q_2 \\ & X_a & \mapsto & X_a \\ & JX_a & \mapsto & -JX_a, \end{array}$$

and

$$\begin{array}{rrcl} \theta: & \mathfrak{g}^\sigma & \to & \mathfrak{g}^\sigma \\ & p_1 & \mapsto & p_1 \\ & q_1 & \mapsto & q_1 \\ & X_a & \mapsto & -X_a. \end{array}$$

The subalgebra of fixed points is $\mathfrak{k} = (\mathfrak{g}^\sigma)^\theta = \mathrm{span}\{p_1, q_1\}$, and we have the chain of totally geodesic submanifolds

$$K \subset \widetilde{G^\sigma}, \qquad G^\sigma = (G/H)^\sigma \subset G/H.$$

Let s be the sign of $b(p)$. We define the left-invariant vector fields $U = 1/(\sqrt{|b(p)|})q_1$, $V = U - s\sqrt{|b(p)|}p_1$ in $\mathfrak{k}$. We thus have $< U, U >= s$, $< V, V >= -s$, $< U, V >= 0$, and $[U, V] = \frac{1}{\sqrt{|b(p)|}}(V - U)$, where $< \, , \, >$ stands for the bilinear form inherited by $\mathfrak{k}$ from $\mathfrak{g}^\sigma$. The Lie algebra $\mathfrak{k}$ is as in Lemma 5.3.5 with $\mu = \frac{1}{\sqrt{|b(p)|}}$, so that K is not geodesically complete.

5.4 Relation with homogeneous plane waves

We exhibit the parallelism between certain kind of (Lorentzian) homogeneous plane waves and Lorentz-Kähler spaces admitting degenerate homogeneous structures of linear type (by Lorentz-Kähler we mean pseudo-Kähler of index 2). Although as far as the author knows there is no formal definition of a plane wave in complex geometry, this relation could allow us to understand the latter spaces as a complex generalization of the former, at least in the important Lorentz-Kähler case, suggesting a starting point for a possible definition of *complex plane wave*.

A *plane wave* is a Lorentz manifold $M = \mathbb{R}^{n+2}$ with metric

$$g = dudv + A_{ab}(u)x^a x^b du^2 + \sum_{a=1}^{n}(dx^a)^2,$$

where $(A_{ab})(u)$ is a symmetric matrix depending on the coordinate u called the *profile* of g. A plane wave is called homogeneous if the Lie algebra of Killing vector fields acts transitively in the tangent space at every point. Among homogeneous plane waves we will be interested in two types. A *Cahen-Wallach space* is defined as a plane wave with profile a constant symmetric matrix (B_{ab}), which makes it symmetric and geodesically complete. On the other hand, a *singular scale-invariant homogeneous plane wave* is a plane wave with profile $(B_{ab})/u^2$, where (B_{ab}) is a constant symmetric matrix. Singular scale-invariant homogeneous plane waves are homogeneous but they are not symmetric. In addition they present a singularity in the cosmological sense at $\{u = 0\}$, since the geodesic deviation equation (or Jacobi equation) governed by the curvature is singular at this set (see [55]). Note that these two kind of spaces are composed by the twisted product of a totally geodesic flat wave front and a 2-dimensional manifold containing time and the direction of propagation. This 2-dimensional space gives the real geometric information of the total manifold and in particular it contains a null parallel vector field. They are all VSI, and the curvature information is contained in the profile $A_{ab}(u)$, since the only non-vanishing component of the curvature is given by

$$R_{uaub} = -A_{ab}(u), \qquad a, b = 1, \ldots, n.$$

Cahen-Wallach spaces are one of the possible indecomposable simply-connected Lorentzian symmetric spaces together with $(\mathbb{R}, -dt^2)$, the de Sitter, and the anti de Sitter spaces (see [15]). On the other hand, in [46] the following characterization is given.

Theorem 5.4.1 *Let (M, g) be a connected pseudo-Riemannian manifold of dimension $n+2$ admitting a degenerate homogeneous pseudo-Riemannian structure of linear type (see Subsection 4.2.1) with $g(\xi, \xi) = 0$. Then (M, g) is locally isometric to $\mathbb{R}^{n+2}$ with metric*

$$ds^2 = dudv + \frac{B_{ab}}{u^2} x^a x^b du^2 + \sum_{a=1}^{n} \varepsilon_a (dx^a)^2$$

for some symmetric matrix (B_{ab}) and $\varepsilon_a = \pm 1, a = 1, \ldots, n$.

Note that for Lorentzian signature this means that a manifold admitting a degenerate homogeneous structure of linear type is locally a singular scale-invariant homogeneous plane wave. Conversely, it is easy to see that every singular scale-invariant homogeneous plane wave admits such a homogeneous structure with $\xi = -\frac{1}{u}\partial_v$.

In the Lorentz-Kähler case, according to [33] there is only one possibility for a simply-connected, indecomposable (and not irreducible), symmetric space of complex dimension 2 and signature $(2, 2)$ with a null parallel complex vector field, that is a manifold with holonomy

$$\mathfrak{hol}_{n=0}^{\gamma_1=0,\gamma_2=0} = \mathbb{R} \begin{pmatrix} i & i \\ -i & -i \end{pmatrix}$$

in the notation of [33]. In order to get a plane wave structure we add a plane wave front by considering a manifold (M, g, J) with holonomy $\mathfrak{hol}_{n=0}^{\gamma_1=0,\gamma_2=0} \oplus \{0_n\}$. Note that this is the holonomy algebra in Proposition 5.2.4 for $\epsilon = -1$.

Proposition 5.4.2 *Let (M, g, J) be a locally symmetric Lorentz-Kähler manifold of dimension $2n+4$, $n \geq 0$, with holonomy $\mathfrak{hol}_{n=0}^{\gamma_1=0,\gamma_2=0} \oplus \{0_{\mathrm{n}}\}$. Then the metric g is locally of the form*

$$g = dw^1 dz^1 + dw^2 dz^2 + b(dw^1 dw^1 + dw^2 dw^2) + \sum_{a=1}^{n} (dx^a dx^a + dy^a dy^a), \qquad (5.29)$$

where the function b only depends on w^1 and w^2 and satisfies

$$\Delta b = b_0, \quad b_0 \in \mathbb{R} - \{0\}.$$

Proof. Looking at the holonomy representation, there are two parallel isotropic (real) vector fields Z and JZ on M. Let $\alpha^1 = g(\cdot, Z)$ and $\alpha^2 = -\alpha^1 \circ J$, consider the complex form $\alpha = \alpha^1 + i\alpha^2$. Since $\nabla Z = 0 = \nabla JZ$, we have $\nabla \alpha = 0$, hence in particular α is holomorphic and closed. This means that locally there is a holomorphic function $w : U \to \mathbb{C}$ such that $dw = \alpha$. Since dw is non-zero at some point and it is parallel, we have that dw is nowhere vanishing. Hence if the set $w^{-1}(\lambda)$, $\lambda \in \mathbb{C}$, is non-empty then it defines a complex hypersurface in U. Let $p \in M$ and let $q_1 \in T_pM$ be such that $g(Z_p, q_1) = 1$ with $g(q_1, q_1) \neq 0$. The subspace $E = \mathrm{Span}\{Z_p, JZ_p, q_1 Jq_1\}$ is invariant by holonomy, hence so is $E^\perp$. In fact, the holonomy action is trivial on $E^\perp$. This implies that there are parallel vector fields E_i, JE_i, $i = 1, \ldots, n$, which are an orthonormal basis of $E^\perp$ at every point. In addition, it is easy to see that Z, JZ, E_i, JE_i are always tangent to the hypersurfaces $w^{-1}(\lambda)$. We can thus take coordinates $\{w^1, w^2, z^1, z^2, x^a, y^a\}$ with $w = w^1 + iw^2$, $\partial_{z^1} = Z$, $\partial z^2 = JZ$, $\partial_{x^i} = E_i$ and $\partial_{y^i} = JE_i$, and such that

$$g = dw^1 dz^1 + dw^2 dz^2 + b(dw^1 dw^1 + dw^2 dw^2) + \sum_{a=1}^{n} (dx^a dx^a + dy^a dy^a).$$

In addition, the only non-zero element of the curvature tensor field is

$$R_{\partial w^1 \partial w^2 \partial w^1 \partial w^2} = \frac{1}{2} \Delta b,$$

where Δ stands for the Laplace operator with respect to the variables (w^1, w^2). The condition of being locally symmetric is then

$$\nabla R = 0 \quad \Leftrightarrow \quad \Delta b = b_0,$$

for $b_0 \in \mathbb{R} - \{0\}$. ■

The previous Proposition suggests to consider the pseudo-Kähler manifold $(\mathbb{C}^{2+n}, g)$ with g as in (5.29) as a natural Lorentz-Kähler analogue to Cahen-Wallach spaces. Note that equation $\Delta b = b_0$ admits singular solutions. Nevertheless, as Cahen-Wallach spaces are simply-connected, in order to have an actual analogue we only consider non-singular functions b, so that $(\mathbb{C}^{2+n}, g)$ is complete.

On the other hand, since Lorentzian singular scale-invariant homogeneous plane waves are characterized by degenerate pseudo-Riemannian homogeneous structures of linear type, from Propositions 5.2.5 and 5.2.6, the natural analogues to these spaces are Lorentz-Kähler manifolds with degenerate homogeneous pseudo-Kähler structures of linear type. More precisely, the spaces $(\mathbb{C}^{n+2} - \{\|w\|_\lambda = 0\}, g)$ with $\|w\|_\lambda$ as in (5.24), g as in (5.25) with $\epsilon = -1$ and signature $(2, 2+2n)$, and $(\mathbb{C}^{n+2}, \bar{g})$ with g given in (5.29), are composed by the twisted product of a flat and totally geodesic complex manifold and a 2-dimensional complex manifold containing a null parallel complex vector field. Moreover, expression (5.25) and (5.29) are the same except for the function b, which has a different Laplacian in each case. As a straight forward computation shows, the curvature tensor of both metrics is

$$R = \frac{1}{2} \Delta b (dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2),$$

whence all the curvature information is contained in the Laplacian of the function b. For this reason, analogously to Lorentz plane waves, we call Δb the profile of the metric.

It is worth noting that in the Lorentz case one goes from Cahen-Wallach spaces to singular scale-invariant homogeneous plane waves by making the profile be singular with a term $1/u^2$. Doing so, the space is no longer geodesically complete and a cosmological singularity at $\{u = 0\}$ is created. In the same way, in the Lorentz-Kähler case one goes from metric (5.29) to (5.25) by making the profile be singular with a term $1/\|w\|_\lambda^4$, and again one transforms a geodesically complete space into a geodesically incomplete space creating a singularity at $\{\|w\|_\lambda = 0\}$. Finally, we also note that all these metrics are VSI. This reinforces the parallelism and exhibits a close relation between these two couples of spaces.

	Symmetric space	Deg. homog. of linear type
Lorentz	Cahen-Wallach spaces Profile: $A(u) = B(const.)$ Geodesically complete	Singular s.-i. homog. plane wave Profile: $A(u) = B/u^2$ Geodesically incomplete
Lorentz-Kähler	$\mathbb{C}^{2+n}$ with metric (5.29) Profile: $\Delta b = R_0(const.)$ Geodesically complete	$\mathbb{C}^{2+n} - \{\|w\|_\lambda = 0\}$ with metric (5.25) Profile: $\Delta b = R_0/\|w\|_\lambda^4$ Geodesically incomplete

Table 5.2: Relation between homogeneous plane waves

Chapter 6

Homogeneous ϵ-quaternion Kähler structures of linear type

In this chapter we study homogeneous structures of linear type on pseudo-quaternion Kähler and para-quaternion Kähler manifolds. On the one hand, we obtain that non-degenerate homogeneous pseudo-quaternion and para-quaternion Kähler structures of linear type characterize spaces of constant quaternionic and para-quaternionic sectional curvature. Moreover, if the metric is not definite, we show that the corresponding quaternionic and para-quaternionic space forms locally admit this kind of structures, but unlike in the Riemannian setting, the global existence is faced with the completeness of the metric. On the other hand, we show that pseudo-quaternion and para-quaternion Kähler manifolds admitting degenerate homogeneous pseudo-quaternion and para-quaternion Kähler structures of linear type are flat. This suggests that the notion of homogeneous plane wave cannot be realized in geometries of quaternionic type.

Since many features in the geometry of pseudo-quaternion and para-quaternion Kähler manifolds are very similar, it is very convenient to develop the arguments and the results simultaneously. For this reason we unify this geometries through the notion of ϵ-quaternion Kähler manifold. Let $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$, where $\epsilon_1 = -1$, $\epsilon_2, \epsilon_3 = \pm 1$.

Definition 6.0.3 *Let (M, g) be a pseudo-Riemannian manifold.*

1. *An almost ϵ-quaternion Hermitian structure on (M, g) is a subbundel $Q \subset \mathfrak{so}(TM)$ such that*
$$J_a^2 = \epsilon_a \mathrm{Id}, \quad a = 1, 2, 3, \qquad J_1 J_2 = J_3.$$
2. *(M, g) is called ϵ-quaternion Kähler if it is strongly oriented and admits a parallel almost ϵ-quaternion Hermitian structure with respect to the Levi-Civita connection.*

This way, one recovers the corresponding formula or result in the pseudo-quaternion Kähler and the para-quaternion Kähler cases by substituting $\epsilon = (-1, -1, -1)$ and $\epsilon = (-1, 1, 1)$ respectively. In particular we can write a homogeneous ϵ-quaternion Kähler structure of linear type as

$$S_X Y = g(X, Y)\xi - g(Y, \xi)X - \sum_{a=1}^{3} \epsilon_a \big(g(J_a Y, \xi) J_a X - g(X, J_a Y) J_a \xi\big) + \sum_{a=1}^{3} g(X, \zeta^a) J_a Y, \quad (6.1)$$

for some local vector fields ξ, ζ^a, $a = 1, 2, 3$. The notions of degenerate and non-degenerate structures remain the same. We shall also use the term ϵ-quaternion sectional curvature, which includes the quaternionic and para-quaternionic cases in the obvious way. In addition $\mathbb{H}^\epsilon$ will denote the set of quaternions or para-quaternions, $\mathfrak{sp}^\epsilon(n)$ will denote the Lie algebras $\mathfrak{sp}(p, q)$ with $p + q = n$ (where the signature (p, q) is assumed to be known) or $\mathfrak{sp}(n, \mathbb{R})$, and $Sp^\epsilon(n)$ will denote the Lie groups $Sp(p, q)$ or $Sp(n, \mathbb{R})$ for $\epsilon = (-1, -1, -1)$ and $\epsilon = (-1, 1, 1)$ respectively.

6.1 Characterizing homogeneous ϵ-quaternion Kähler structures of linear type

Theorem 6.1.1 *Let (M, g, Q) be a connected ϵ-quaternion Kähler manifold of dimension $4n \geqslant 8$ admitting a homogeneous ϵ-quaternion Kähler structure of linear type S. If S is non-degenerate, then (M, g, Q) has constant ϵ-quaternion sectional curvature $-4g(\xi, \xi)$, and $\zeta^a = 0$ for $a = 1, 2, 3$. If S is degenerate then (M, g, Q) is flat.*

Proof. By Propositions 1.2.21 and 1.2.25 we decompose the curvature tensor field of (M, g, Q) as $R = \nu_q R^0 + R^{\mathfrak{sp}^\epsilon(n)}$, where $R^{\mathfrak{sp}^\epsilon(n)}$ is a curvature tensor field of type $\mathfrak{sp}^\epsilon(n)$. Recall that the space of algebraic curvature tensors $\mathcal{R}^{\mathfrak{sp}^\epsilon(n)}$ is $[S^4E]$ with $E = \mathbb{C}^{2n}$ for $\epsilon = (-1, -1, -1)$, and S^4E with $E = \mathbb{R}^{2n}$ for $\epsilon = (-1, 1, 1)$. Since R^0 is $Sp^\epsilon(n)Sp^\epsilon(1)$-invariant, the covariant derivative ∇R^0 vanishes. Moreover, for every vector field X, S_X acts as an element of $\mathfrak{sp}^\epsilon(n) + \mathfrak{sp}^\epsilon(1)$, whence $S \cdot R^0 = 0$. Using that $\widetilde{\nabla} R = 0$ and $\widetilde{\nabla} = \nabla - S$ we have

$$0 = \widetilde{\nabla} R = \nu_q \widetilde{\nabla} R^0 + \widetilde{\nabla} R^{\mathfrak{sp}^\epsilon(n)} = \nabla R^{\mathfrak{sp}^\epsilon(n)} - S \cdot R^{\mathfrak{sp}^\epsilon(n)}.$$

Writing $T^*M \otimes (\mathfrak{sp}^\epsilon(n) + \mathfrak{sp}^\epsilon(1)) = T^*M \otimes \mathfrak{sp}^\epsilon(n) + T^*M \otimes \mathfrak{sp}^\epsilon(1)$ we can decompose $S = S_E + S_H$, and hence $S_H \cdot R^{\mathfrak{sp}^\epsilon(n)} = 0$. We thus obtain

$$\nabla R = \nabla R^{\mathfrak{sp}^\epsilon(n)} = S_E \cdot R^{\mathfrak{sp}^\epsilon(n)},$$

which we can write as

$$(\nabla_X R)_{YZWU} = -R^{\mathfrak{sp}^\epsilon(n)}_{S_X YZWU} - R^{\mathfrak{sp}^\epsilon(n)}_{YS_X ZWU} - R^{\mathfrak{sp}^\epsilon(n)}_{YZS_X WU} - R^{\mathfrak{sp}^\epsilon(n)}_{YZWS_X U}. \tag{6.2}$$

Taking the cyclic sum in X, Y, Z and applying Bianchi identities we obtain

$$\begin{aligned} 0 = \mathop{\mathfrak{S}}_{XYZ} \Big\{ & 2g(X,\xi) R^{\mathfrak{sp}^\epsilon(n)}_{YZWU} + g(X,W) R^{\mathfrak{sp}^\epsilon(n)}_{YZ\xi U} + g(X,U) R^{\mathfrak{sp}^\epsilon(n)}_{YZW\xi} \\ & + 2\sum_a \epsilon_a \big(g(X, J_a Y) R^{\mathfrak{sp}^\epsilon(n)}_{J_a\xi ZWU} + g(X, J_a W) R^{\mathfrak{sp}^\epsilon(n)}_{YZJ_a\xi U} + g(X, J_a U) R^{\mathfrak{sp}^\epsilon(n)}_{YZWJ_a\xi} \big) \Big\}. \end{aligned}$$

Contracting the previous formula with respect to X and W, and taking into account that $R^{\mathfrak{sp}^\epsilon(n)}$ is traceless we obtain

$$(4n + 2) R^{\mathfrak{sp}^\epsilon(n)}_{YZ\xi U} = 0,$$

for every vector fields Z, Y, U. Expanding the expression of S in (6.2) and using the previous formula we arrive at

$$0 = \mathop{\mathfrak{S}}_{XYZ} \theta(X) R^{\mathfrak{sp}^\epsilon(n)}_{YZWU},$$

where $\theta = \xi^\flat$, or equivalently

$$0 = \theta \wedge R^{\mathfrak{sp}^\epsilon(n)}_{WU}. \tag{6.3}$$

Noting that $R^{\mathfrak{sp}^\epsilon(n)}$ satisfies the symmetries $R^{\mathfrak{sp}^\epsilon(n)}_{XJ_aYWU} + R^{\mathfrak{sp}^\epsilon(n)}_{J_aXYWU} = 0$, $a = 1, 2, 3$, we also have

$$0 = (\theta \circ J_a) \wedge R^{\mathfrak{sp}^\epsilon(n)}_{WU} = 0, \qquad a = 1, 2, 3. \tag{6.4}$$

It is easy to see that a curvature tensor of type $\mathfrak{sp}^\epsilon(n)$ satisfying equations (6.3) and (6.4) must vanish. Therefore we conclude that $R = \nu_q R^0$.

Now, making use of $\widetilde{\nabla} S = 0$ together with (6.1), and taking into account (4.6) and (4.11) we have that

$$0 = g(X,Y)\widetilde{\nabla}_Z\xi - g(\widetilde{\nabla}_Z\xi,Y)X - \sum_a \epsilon_a(g(\widetilde{\nabla}_Z\xi,J_aY)J_aX + g(X,J_aY)J_a\widetilde{\nabla}_Z\xi) \\ + \sum_a g(X,\widetilde{\nabla}_Z\zeta^a - \sum_b d_{ba}(Z)\zeta^b)J_aY,$$

where (d_{ba}) is a matrix of 1-forms sitting in $\mathfrak{sp}^\epsilon(1)$. Taking $X \in (\mathbb{H}^\epsilon\xi)^\perp$ with $g(X,X) \neq 0$, and multiplying by X in the previous formula we obtain that

$$\widetilde{\nabla}_Z\xi = 0. \tag{6.5}$$

Hence

$$\widetilde{\nabla}_Z\zeta^a = \sum_b d_{ba}\zeta^b, \qquad a = 1,2,3. \tag{6.6}$$

From (6.5), (4.6) and (4.11) we compute

$$\nabla_X J_a\xi = \sum_b d_{ab}(X)J_b\xi + g(X,J_a\xi)\xi \\ - \sum_b \epsilon_b(g(\xi,J_bJ_a\xi)J_bX - g(X,J_bJ_aX)J_b\xi) + \sum_b g(X,\zeta^b)J_bJ_a\xi. \tag{6.7}$$

On the other hand

$$\begin{aligned} R_{XY}\xi &= -\nabla_X\nabla_Y\xi + \nabla_Y\nabla_X\xi + \nabla_{[X,Y]}\xi \\ &= -g(Y,\nabla_X\xi)\xi - g(Y,\xi)\nabla_X\xi + g(X,\nabla_Y\xi)\xi + g(X,\xi)\nabla_Y\xi \\ &\quad - \sum_a \epsilon_a\big(g(Y,\nabla_XJ_a\xi)J_a\xi + g(Y,J_a\xi)\nabla_XJ_a\xi \\ &\qquad\qquad - g(X,\nabla_YJ_a\xi)J_a\xi - g(X,J_a\xi)\nabla_YJ_a\xi\big) \\ &\qquad + \sum_a -g(Y,\nabla_X\zeta^a)J_a\xi - g(Y,\zeta^a)\nabla_XJ_a\xi \\ &\qquad + g(X,\nabla_Y\zeta^a)J_a\xi + g(X,\zeta^a)\nabla_YJ_a\xi. \end{aligned} \tag{6.8}$$

If S is non-degenerate, taking $X, Y \in (\mathbb{H}^\epsilon\xi)^\perp$, we have $g(R_{XY}\xi,X) = 0$ from $R = \nu_q R^0$ on the one hand, and

$$g(R_{XY}\xi,X) = \sum_a g(X,\zeta^a)g(\xi,\xi)g(J_aY,X)$$

from (6.8) on the other. Moreover, for $Y = J_bX$ it reduces to

$$g(R_{XJ_bX}\xi,X) = -\epsilon_b g(\xi,\xi)g(X,\zeta^b)g(X,X).$$

This implies that $g(X,\zeta^b) = 0$, so that

$$\zeta^b \in \mathbb{H}^\epsilon\xi, \qquad b = 1,2,3.$$

Recalling (6.5) we have that $g(\xi,\nabla_Y\xi) = 0$. Applying this and (4.6) and (4.11) to (6.8) with $X = \xi$ and $Y \in (\mathbb{H}^\epsilon\xi)^\perp$ we obtain

$$g(Y,\nabla_YJ_a\xi) = 0, \qquad g(Y,\nabla_Y\zeta^a) = 0, \qquad g(\xi,\nabla_Y\zeta^a) = 0,$$

$$g(Y,\nabla_\xi J_a\xi) = g(Y,J_a\nabla_\xi\xi) + \sum_b g(Y,b_{ab}(\xi)J_b\xi) = 0,$$

$$g(Y, \nabla_Y J_a \xi) = g(\xi, J_a \nabla_Y \xi) + \sum_b g(\xi, b_{ab} J_b \xi) = 0.$$

Hence

$$\begin{aligned}
R_{\xi Y}\xi &= g(\xi,\xi)\nabla_Y \xi + \sum_a g(\xi,\zeta^a)\nabla_Y J_a \xi \\
&= -g(\xi,\xi)^2 Y - \sum_a g(\xi,\zeta^a)\sum_b \epsilon_b g(J_b J_a \xi, \xi) J_b Y \\
&= -g(\xi,\xi)^2 Y - \sum_a g(\xi,\zeta^a) g(\xi,\xi) J_a Y.
\end{aligned}$$

Comparing with $R_{\xi Y}\xi = \nu_q R^0_{\xi Y}\xi = \nu_q g(\xi,\xi)Y$ we deduce that $\nu_q = -g(\xi,\xi)$ and $g(\xi,\zeta^a) = 0$. Finally we take again $X, Y \in (\mathbb{H}^\epsilon \xi)^\perp$ in (6.8) obtaining

$$\begin{aligned}
R_{XY}\xi = &-g(Y,\nabla_X \xi)\xi + g(X,\nabla_Y \xi)\xi \\
&- \sum_a \epsilon_a (g(Y,\nabla_X J_a \xi) J_a \xi - g(X, \nabla_Y J_a \xi) J_a \xi) \\
&+ \sum_a -g(Y, \nabla_X \zeta^a) J_a \xi + g(X, \nabla_Y \zeta^a) J_a \xi.
\end{aligned}$$

Taking into account (6.6), the previous formula reads

$$R_{XY}\xi = 2\sum_a \epsilon_a g(Y, J_a X) g(\xi,\xi) J_a \xi + 2 \sum_{a,b} \epsilon_b g(Y, J_b X) g(\xi, J_b \zeta^a) J_a \xi,$$

and comparing with

$$R_{XY}\xi = \nu_q R^0_{XY}\xi = -2\sum_a \epsilon_a g(\xi,\xi) g(X, J_a Y) J_a \xi$$

we have

$$g(J_b \zeta^a, \xi) = 0, \qquad a, b = 1, 2, 3.$$

This in conjunction with $\zeta^a \in (\mathbb{H}^\epsilon \xi)^\perp$ and $g(\zeta^a, \xi) = 0$ gives

$$\zeta^a = 0, \qquad a = 1, 2, 3.$$

Finally, if S is degenerate, we apply $\widetilde{\nabla}\xi = 0$ and $\nabla J_a = \sum_b c_{ab} J_b$ to (6.8) obtaining that $R_{XY}\xi \in \operatorname{span}\{\xi, J_1\xi, J_2\xi, J_3\xi\}$. Comparing this with $R_{XY}\xi = \nu_q R^0_{XY}\xi$ we deduce that $\nu_q = 0$, so that $R = 0$. ∎

Remark 6.1.2 *In the non-degenerate case, for $\epsilon = (-1,-1,-1)$, if $g(\xi,\xi) > 0$ then $c = -4g(\xi,\xi) < 0$, so that spaces with negative definite metric and constant negative quaternionic sectional curvature cannot admit non-degenerate homogeneous pseudo-quaternion structures of linear type. Similarly, if $g(\xi,\xi) < 0$ then $c > 0$, so that spaces with positive definite metric and constant positive quaternionic sectional curvature are also excluded.*

Remark 6.1.3 *For $\epsilon = -1$, if $g(\xi,\xi) > 0$ then $c = -4g(\xi,\xi) < 0$, so that spaces with negative definite metric and constant negative holomorphic sectional curvature cannot admit non-degenerate homogeneous pseudo-Kähler structures of linear type. Similarly, if $g(\xi,\xi) < 0$ then $c > 0$, so that spaces with positive definite metric and constant positive holomorphic sectional curvature are also excluded.*

From the arguments used in the proof of Theorem 6.1.1, the following result is evident.

Proposition 6.1.4 *Let (M, g, J_1, J_2, J_3) be a pseudo-hyper-Kähler (resp. para-hyper-Kähler) manifold admitting a homogeneous pseudo-hyper-Kähler (para-hyper-Kähler) structure of linear type. Then (M, g, J_1, J_2, J_3) is flat.*

6.2 Infinitesimal models, homogeneous models and completeness

Recall the definition of infinitesimal model, transvection algebra and homogeneous model associated to a homogeneous structure (Section 2.3). Since by Theorem 6.1.1, degenerate homogeneous ϵ-quaternion Kähler structures of linear type only exist for flat metrics, we will be only interested in non-degenerate structures. The aim of this section is to prove the following results.

Theorem 6.2.1 *With the exception of* $\mathbb{H}\mathrm{P}^n_0$ *and* $\mathbb{H}\mathrm{H}^n_n$*, the indefinite* ϵ*-quaternion space forms* $\mathbb{H}\mathrm{P}^n_s$*,* $\mathbb{H}\mathrm{H}^n_s$ *and* $\widetilde{\mathbb{H}}\mathrm{P}^n$ *locally admit a non-degenerate homogeneous* ϵ*-quaternion Kähler structure of linear type.*

Theorem 6.2.2 *Let* (M, g, Q) *be a connected and simply-connected* ϵ*-quaternion Kähler manifold with* $\dim M \geqslant 8$ *admitting a non-degenerate homogeneous* ϵ*-quaternion Kähler structure of linear type. If* g *is not definite then* (M, g, Q) *is not complete.*

Remark 6.2.3 *Note that Theorem 6.2.2 implies that the indefinite* ϵ*-quaternion space forms* $\mathbb{H}\mathrm{P}^n_p$*,* $\mathbb{H}\mathrm{H}^n_p$ *and* $\widetilde{\mathbb{H}}\mathrm{P}^n$ *do not admit a globally defined non-degenerate homogeneous* ϵ*-quaternion Kähler structure of linear type except for* $\mathbb{H}\mathrm{P}^n_n$ *and* $\mathbb{H}\mathrm{H}^n_0$*.*

In order to prove Theorems 6.2.1 and 6.2.2, we can adapt in a straightforward way the procedure used to prove Theorems 5.3.1 and 5.3.2 in Section 5.3. In fact, the only difference is that, by Remark 1.2.22, we will need only consider the spaces $\mathbb{H}\mathrm{H}^n_p$ and $\widetilde{\mathbb{H}}\mathrm{P}^n$, since by Theorem 5.1.2 our spaces of linear type are locally $\pm$-isometric to one of these models. In addition, the expression of this spaces as *Isom*/*Isot* are given by (1.4) and (1.6) respectively. We thus just have to specify all the components involved in that procedure for each case. Due to differences we treat them separately.

6.2.1 The para-quaternion Kähler case

During this subsection $\widetilde{\mathbb{H}}$ denotes the set of para-quaternions with imaginary units i, j, k. Using (6.1) we compute

$$
\begin{aligned}
R^S_{XY}W = -g(\xi,\xi)\Big\{ & g(X,W)Y - g(Y,W)X \\
& + \sum_a \epsilon_a (g(X, J_a W) J_a Y - g(Y, J_a W) J_a X)\Big\} \\
& - 2\sum_a \epsilon_a (g(\xi, J_a W) g(X, J_a Y)\xi + g(\xi, W) g(X, J_a Y) J_a \xi) \\
& + 2\sum_a (g(X, J_a Y) g(\xi, J_c W) J_b \xi - g(X, J_a Y) g(\xi, J_b W) J_c \xi),
\end{aligned}
$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$, and $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, 1, 1)$. From $\widetilde{R} = R - R^S$ and $R = -g(\xi, \xi) R^0$, we obtain

$$
\begin{aligned}
\widetilde{R}_{XY}W = & -2\sum_a \epsilon_a g(\xi,\xi) g(X, J_a Y) J_a W \\
& + 2\sum_a g(X, J_a Y)\big(\epsilon_a g(\xi, J_a W)\xi + \epsilon_a g(\xi, W) J_a \xi - g(\xi, J_c W) J_b \xi \\
& \qquad + g(\xi, J_b W) J_c \xi\big).
\end{aligned}
$$

In particular

$$
\begin{aligned}
\widetilde{R}_{XY}\xi &= 0,\\
\widetilde{R}_{XY}J_1\xi &= 4g(\xi,\xi)\big(g(X,J_2Y)J_3\xi - g(X,J_3Y)J_2\xi\big),\\
\widetilde{R}_{XY}J_2\xi &= 4g(\xi,\xi)\big(g(X,J_1Y)J_3\xi - g(X,J_3Y)J_1\xi\big),\\
\widetilde{R}_{XY}J_3\xi &= 4g(\xi,\xi)\big(g(X,J_2Y)J_1\xi - g(X,J_1Y)J_2\xi\big),\\
\widetilde{R}_{XY}Z &= -2g(\xi,\xi)\sum_a \epsilon_a g(X,J_aY)J_aZ, \quad \text{for } Z \in (\widetilde{\mathbb{H}}\xi)^\perp.
\end{aligned}
$$

This shows that $\mathfrak{hol}^{\widetilde{\nabla}}$ acts on

$$T_pM = \mathbb{R}\xi + \operatorname{Im}\widetilde{\mathbb{H}}\xi + (\widetilde{\mathbb{H}}\xi)^\perp$$

as $\mathfrak{sp}(1,\mathbb{R})$ acts on the representation

$$\mathbb{R} + \mathfrak{sp}(1,\mathbb{R}) + EH,$$

where $E = \mathbb{R}^{2n-2}$ and $H = \mathbb{R}^2$. In addition, for $Y \in (\widetilde{\mathbb{H}}X)^\perp$ we have $\widetilde{R}_{XY} = 0$, and for X such that $g(X,X) = 1/(2g(\xi,\xi))$ we have

$$
\begin{gathered}
\widetilde{R}_{XJ_aX}\xi = 0, \qquad \widetilde{R}_{XJ_aX}J_b\xi = -[J_a,J_b]\xi \qquad \text{and}\\
\widetilde{R}_{XJ_aX}Z = -J_aZ, \quad Z \in (\widetilde{\mathbb{H}}\xi)^\perp.
\end{gathered}
$$

Denoting by $\mathcal{J}_a$ the element of $\mathfrak{hol}^{\widetilde{\nabla}}$ that acts as J_a on $(\widetilde{\mathbb{H}}\xi)^\perp$, the remaining brackets of $\mathfrak{g}$ are

$$[Z_1,Z_2] = 2\sum_a \epsilon_a g(Z_1,J_aZ_2)(J_a\xi - g(\xi,\xi)\mathcal{J}_a), \tag{6.9}$$

$$[\xi,Z] = g(\xi,\xi)Z, \tag{6.10}$$

$$[J_a\xi,Z] = g(\xi,\xi)J_aZ, \tag{6.11}$$

$$[\xi,J_a\xi] = 2g(\xi,\xi)J_a\xi - 2g(\xi,\xi)^2\mathcal{J}_a, \tag{6.12}$$

$$[J_a\xi,J_b\xi] = \epsilon_c\left(4g(\xi,\xi)J_c\xi - 2g(\xi,\xi)^2\mathcal{J}_c\right), \tag{6.13}$$

for (a,b,c) any cyclic permutation of $(1,2,3)$, where $Z, Z_1, Z_2 \in (\widetilde{\mathbb{H}}X)^\perp$. The transvection algebra is thus

$$\mathfrak{g} = T_pM + \mathfrak{hol}^{\widetilde{\nabla}} \cong \mathbb{R}\xi + \operatorname{Im}\widetilde{\mathbb{H}}\xi + (\widetilde{\mathbb{H}}\xi)^\perp + \mathfrak{sp}(1,\mathbb{R}),$$

where $\mathfrak{hol}^{\widetilde{\nabla}}$ acts on T_pM as $\mathfrak{sp}(1,\mathbb{R})$ acts on the representation $\mathbb{R} + \mathfrak{sp}(1,\mathbb{R}) + \widetilde{\mathbb{H}}^{n-1}$. We now identify this algebra with a subalgebra of $\mathfrak{sp}(n+1,\mathbb{R})$. The algebra $\mathfrak{sp}(n+1,\mathbb{R}) = \{\, A \in \mathfrak{gl}(n+1,\widetilde{\mathbb{H}}) \mid A + A^* = 0 \,\}$ has Cartan decomposition

$$\mathfrak{sp}(n+1,\mathbb{R}) = \mathfrak{sp}(n,\mathbb{R}) + \mathfrak{sp}(1,\mathbb{R}) + \mathfrak{p},$$

where

$$
\begin{aligned}
\mathfrak{sp}(n,\mathbb{R}) + \mathfrak{sp}(1,\mathbb{R}) &= \left\{ \begin{pmatrix} A & 0 \\ 0 & q \end{pmatrix} \,\middle|\, A \in \mathfrak{sp}(n,\mathbb{R}), q \in \operatorname{Im}\widetilde{\mathbb{H}} \right\},\\
\mathfrak{p} &= \left\{ \begin{pmatrix} 0 & v \\ -v^* & 0 \end{pmatrix} \,\middle|\, v \in \widetilde{\mathbb{H}}^n \right\}.
\end{aligned}
$$

The maximal abelian subalgebra of $\mathfrak{p}$ is up to isomorphism $\mathfrak{a} = \operatorname{Span}\{A_0\}$, where

$$A_0 = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & 0 & j \\ 0 & j & 0 \end{pmatrix}.$$

The restricted roots are $\{\pm\lambda, \pm 2\lambda\}$, where $\lambda(A_0) = 1$. With the choice of positive roots $\{\lambda, 2\lambda\}$, the corresponding root spaces are

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} 0_{n-1} & -vj & v \\ -j\bar{v} & 0 & 0 \\ -\bar{v} & 0 & 0 \end{pmatrix} \middle| \, v \in \widetilde{\mathbb{H}}^{n-1} \right\},$$
$$\mathfrak{n}_2 = \left\{ \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & -jqj & jq \\ 0 & \bar{q}j & q \end{pmatrix} \middle| \, q \in \operatorname{Im} \widetilde{\mathbb{H}} \right\}.$$

Therefore, the algebra $\mathfrak{sp}(n+1, \mathbb{R})$ decomposes as

$$\mathfrak{sp}(n+1, \mathbb{R}) = \mathfrak{sp}(n, \mathbb{R}) + \mathfrak{sp}(1, \mathbb{R}) + \mathfrak{a} + \mathfrak{n}_1 + \mathfrak{n}_2.$$

We consider the ad-invariant complement $\mathfrak{m}_\lambda = \mathfrak{a} + \mathfrak{n}_1 + \mathfrak{p}_\lambda$ where

$$\mathfrak{p}_\lambda = \left\{ \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & (\lambda - 1)jqj & jq \\ 0 & \bar{q}j & (\lambda + 1)q \end{pmatrix} \middle| \, q \in \operatorname{Im} \widetilde{\mathbb{H}} \right\}$$

and $\lambda \in \mathbb{R}$. From the brackets (6.9)–(6.13) we see that $\xi \in \mathfrak{a}$ and $J_a\xi \in \mathfrak{p}_\lambda$, and by the holonomy action we identify $\mathfrak{n}_1$ with $(\widetilde{\mathbb{H}}\xi)^\perp$. In addition, comparing the brackets

$$[Z_1, Z_2] = -2 \sum_a \epsilon_a g(J_a Z_1, Z_2)(J_a\xi - g(\xi, \xi)\mathcal{J}_a),$$

and

$$\begin{aligned}[v, w] = 2\langle v(-i), w\rangle \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & i & -k \\ 0 & -k & i \end{pmatrix} - 2\langle v(-j), w\rangle \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & -j & 1 \\ 0 & -1 & j \end{pmatrix} \\ - 2\langle v(-k), w\rangle \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & k & -i \\ 0 & -i & k \end{pmatrix},\end{aligned}$$

where $v, w \in \mathfrak{n}_1 \cong (\widetilde{\mathbb{H}}\xi)^\perp \cong \widetilde{\mathbb{H}}^{n-1}$ (as para-quaternion vector spaces) and $\langle v, w\rangle = \operatorname{Re}(v^* w)$, we have

$$J_1\xi - g(\xi, \xi)\mathcal{J}_1 = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & i & -k \\ 0 & -k & i \end{pmatrix}, \quad J_2\xi - g(\xi, \xi)\mathcal{J}_2 = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & -j & 1 \\ 0 & -1 & j \end{pmatrix},$$
$$J_3\xi - g(\xi, \xi)\mathcal{J}_3 = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & k & -i \\ 0 & -i & k \end{pmatrix}.$$

Hence $\mathcal{J}_1$ acts on $\mathfrak{n}_1$ as right multiplication by $-i$, etc., that is

$$\mathcal{J}_1 = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix} \quad \mathcal{J}_2 = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j \end{pmatrix}, \quad \mathcal{J}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & k \end{pmatrix}.$$

The Lie algebra spanned by $\{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$ exponentiates to a closed subgroup of $Sp(n+1, \mathbb{R})$.

Regarding the Lie algebra involutions involved in the proof of Theorem 6.2.2 we take $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ given by

$$\mathcal{J}_1 \mapsto -\mathcal{J}_1, \quad \mathcal{J}_2 \mapsto \mathcal{J}_2, \quad \mathcal{J}_3 \mapsto -\mathcal{J}_3,$$
$$\xi \mapsto \xi, \quad J_1\xi \mapsto -J_1\xi, \quad J_2\xi \mapsto J_2\xi, \quad J_3\xi \mapsto -J_3\xi,$$
$$v_1 + iv_2 + jv_3 + kv_4 \mapsto v_1 - iv_2 + jv_3 - kv_4,$$

for $v_1 + iv_2 + jv_3 + kv_4 \in (\widetilde{\mathbb{H}}\xi)^\perp$. We then let $\tau\colon \mathfrak{g}^\sigma \to \mathfrak{g}^\sigma$ be

$$\mathcal{J}_2 \mapsto -\mathcal{J}_2, \quad \xi \mapsto \xi, \quad J_2\xi \mapsto -J_2\xi, \quad v_1 + iv_2 \mapsto -v_1 + jv_2,$$

and additionally define $\lambda\colon (\mathfrak{g}^\sigma)^\tau \to (\mathfrak{g}^\sigma)^\tau$ by

$$\xi \mapsto \xi, \qquad (v_1 j, \dots, v_{n-2}j, v_{n-1})^T \mapsto (-v_1 j, \dots, -v_{n-2}j, +v_{n-1}j)^T,$$

The fixed point set of the sequence σ, τ, λ is

$$\mathfrak{k} = \mathrm{Span}\{\xi, (0, \dots, 0, j)\},$$

so that the chain of totally geodesic submanifolds is

$$K \subset (G^\sigma)^\tau = (G^\sigma/H^\sigma)^\tau \subset G^\sigma/H^\sigma = (G/H)^\sigma \subset G/H.$$

Once again it is easy to see that K is as in Lemma 5.3.4.

6.2.2 The pseudo-quaternion Kähler case

Throughout this section i, j, k are the imaginary units of the quaternions $\mathbb{H}$. With the help of formula (6.1) we compute

$$\begin{aligned} R^S_{XY}W = -g(\xi,\xi)\Big\{ & g(X,W)Y - g(Y,W)X \\ & + \sum_a (-g(X,J_aW)J_aY + g(Y,J_aW)J_aX)\Big\} \\ & - 2\sum_a \{g(\xi,J_aW)g(X,J_aY)\xi + g(\xi,W)g(X,J_aY)J_a\xi\} \\ & + 2\sum_a \{g(X,J_aY)g(\xi,J_cW)J_b\xi - g(X,J_aY)g(\xi,J_bW)J_c\xi\}, \end{aligned}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. Then $\widetilde{R} = R - R^S$ gives

$$\begin{aligned} \widetilde{R}_{XY}W = & -2g(\xi,\xi)\sum_a g(J_aX,Y)J_aW \\ & + 2\sum_a \{g(\xi,J_aW)g(X,J_aY)\xi + g(\xi,W)g(X,J_aY)J_a\xi\} \\ & - 2\sum_a \{g(X,J_aY)g(\xi,J_cW)J_b\xi - g(X,J_aY)g(\xi,J_bW)J_c\xi\}. \end{aligned}$$

In particular

$$\begin{aligned} \widetilde{R}_{XY}\xi &= 0, \\ \widetilde{R}_{XY}J_1\xi &= -4g(\xi,\xi)\{g(J_3X,Y)J_2\xi - g(J_2X,Y)J_3\xi\}, \\ \widetilde{R}_{XY}J_2\xi &= -4g(\xi,\xi)\{g(J_1X,Y)J_3\xi - g(J_3X,Y)J_1\xi\}, \\ \widetilde{R}_{XY}J_3\xi &= -4g(\xi,\xi)\{g(J_2X,Y)J_1\xi - g(J_1X,Y)J_2\xi\}, \\ \widetilde{R}_{XY}Z &= -2g(\xi,\xi)\sum_a g(J_aX,Y)J_aZ, \qquad \text{for } Z \in (\mathbb{H}\xi)^\perp. \end{aligned}$$

This implies that $\mathfrak{hol}^{\widetilde{\nabla}}$ acts over T_pM as $\mathfrak{sp}(1)$ in the representation

$$T_pM = \mathbb{R}\xi + \mathrm{Im}\,\mathbb{H}\xi + (\mathbb{H}\xi)^\perp = \mathbb{R} + \mathfrak{sp}(1) + [EH],$$

where here $E = \mathbb{C}^{n-1}$. In addition, for $Y \in (\mathbb{H}\xi)^\perp$ we have $\widetilde{R}_{XY} = 0$, and for X such that $g(X,X) = 1/(2g(\xi,\xi))$ we have

$$\widetilde{R}_{XJ_aX}\xi = 0, \quad \widetilde{R}_{XJ_aX}J_bX = -[J_a,J_b]\xi, \quad \widetilde{R}_{XJ_aX}Z = -J_aZ.$$

We denote by $\mathcal{J}_a$ the element $-\widetilde{R}_{XJ_aX}$, which acts as J_a on the factor $[EH] \cong (\mathbb{H}\xi)^\perp$. The remaining brackets of $\mathfrak{g}$ are given by

$$[Z_1,Z_2] = 2\sum_a \{g(J_aZ_1,Z_2)J_a\xi - g(\xi,\xi)g(J_aZ_1,Z_2)\mathcal{J}_a\}, \tag{6.14}$$

$$[\xi,Z] = g(\xi,\xi)Z, \tag{6.15}$$

$$[\xi,J_a\xi] = 2g(\xi,\xi)J_a\xi - 2g(\xi,\xi)^2\mathcal{J}_a, \tag{6.16}$$

$$[J_a\xi,Z] = g(\xi,\xi)J_aZ, \tag{6.17}$$

$$[J_a\xi,J_b\xi] = 4g(\xi,\xi)J_c\xi - 2g(\xi,\xi)^2\mathcal{J}_c, \tag{6.18}$$

for $Z, Z_1, Z_2 \in (\mathbb{H}\xi)^\perp$ and each cyclic permutation (a,b,c) of $(1,2,3)$. The transvection algebra is thus

$$\mathfrak{g} = T_pM + \mathfrak{hol}^{\widetilde{\nabla}} = \mathbb{R}\xi + \operatorname{Im}\mathbb{H}\xi + (\mathbb{H}\xi)^\perp + \mathfrak{sp}(1),$$

where $\mathfrak{hol}^{\widetilde{\nabla}}$ acts on T_pM as $\mathfrak{sp}(1)$ on $\mathbb{R} + \mathfrak{sp}(1) + \mathbb{H}^{n-1}$. Recalling description (1.4) of $\mathbb{H}\mathrm{H}^n_s$ as a symmetric space, we identify $\mathfrak{g}$ with a subalgebra of $\mathfrak{sp}(n-s, s+1)$. The Riemannian case $\mathbb{H}\mathrm{H}^n_0$ is studied in [16], for that reason we suppose $s > 0$. We can also suppose $n - 2s - 1 > 0$ for the sake of simplicity. Let

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma = \operatorname{diag}\big((1)^{n-2s-1}, (\varepsilon)^{s+1}\big),$$

we have that

$$\mathfrak{sp}(n-s,s+1) = \{A \in \mathfrak{gl}(n+1,\mathbb{H}) / \, A^*\Sigma + \Sigma A = 0\}.$$

The algebra $\mathfrak{sp}(\mathfrak{n}-\mathfrak{s},\mathfrak{s}+\mathfrak{1})$ decomposes as

$$\mathfrak{sp}(n-s,s+1) = \mathfrak{sp}(n-s,s) + \mathfrak{sp}(1) + \mathfrak{a} + \mathfrak{n}_1 + \mathfrak{n}_2,$$

where $\mathfrak{a}$ is generated by $A_0 = \operatorname{diag}(0,\dots,0,1,-1)$ and

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} 0_{n-1} & 0 & v \\ -(\Sigma' v)^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, v \in \mathbb{H}^{n-1} \right\}, \quad \mathfrak{n}_2 = \left\{ \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, b \in \operatorname{Im}\mathbb{H} \right\},$$

with $\Sigma' = \operatorname{diag}\big((1)^{n-2s-1}, (\varepsilon)^s\big)$. As in the para-quaternion Kähler case we identify $\mathfrak{n}_1$ with $(\mathbb{H}\xi)^\perp$, and from the matrix expression of $\mathfrak{n}_1$ we obtain

$$J_1\xi - g(\xi,\xi)\mathcal{J}_1 = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{etc.}$$

In addition we have

$$\mathcal{J}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j \end{pmatrix}, \quad \mathcal{J}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix},$$

so that the Lie algebra spanned by $\{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$ gives a closed Lie subgroup of $Sp(n-s,s+1)$.

For the Lie algebra involutions involved in the proof of Theorem 6.2.2 we finally take $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ given by

$$\begin{gathered}\mathcal{J}_1 \mapsto \mathcal{J}_1, \quad \mathcal{J}_2 \mapsto -\mathcal{J}_2, \quad \mathcal{J}_3 \mapsto -\mathcal{J}_3\\ \xi \mapsto \xi, \quad J_1\xi \mapsto J_1\xi, \quad J_2\xi \mapsto -J_2\xi, \quad J_3\xi \mapsto -J_3\xi\\ v_1 + iv_2 + jv_3 + kv_4 \mapsto v_1 + iv_2 - jv_3 - kv_4,\end{gathered}$$

for $v_1 + iv_2 + jv_3 + kv_4 \in (\mathbb{H}\xi)^\perp$. Then we put $\tau\colon \mathfrak{g}^\sigma \to \mathfrak{g}^\sigma$ to be

$$\mathcal{J}_1 \mapsto -\mathcal{J}_1, \quad \xi \mapsto \xi, \quad J_1\xi \mapsto -J_1\xi, \quad v_1 + iv_2 \mapsto v_1 - iv_2,$$

and define $\lambda\colon (\mathfrak{g}^\sigma)^\tau \to (\mathfrak{g}^\sigma)^\tau$ by

$$\xi \mapsto \xi, \qquad (v_1, \dots, v_{n-2}, v_{n-1})^T \mapsto (-v_1, \dots, -v_{n-1}, -v_{n-2})^T.$$

This leads to the chain of totally geodesic submanifolds

$$K = ((G^\sigma)^\tau)^\lambda \subset (G^\sigma)^\tau = (G^\sigma/H^\sigma)^\tau \subset G^\sigma/H^\sigma = (G/H)^\sigma \subset G/H,$$

with K as in Lemma 5.3.4, and so incomplete.

Chapter 7

Reduction of homogeneous structures

Symmetries represent a classical tool in reduction schemes intimately related with different topics as systems of differential equations, variational principles, symplectic or other geometric structures, etc. In particular, reduction is recurrently applied in the setting of homogeneous spaces. The goal of this chapter is the study of the behaviour of homogeneous structures by reduction under subgroups of the group of isometries. In particular, this gives rise to new homogeneous structures in the orbit space of the action. Additionally, the reduction process reveals and sheds light to some previously known properties of some homogeneous structures. Finally, this technique allows to obtain information about homogeneous structures in the unreduced space from homogeneous structures in the orbit space.

Let $\pi : \bar{M} \to M$ be an H-principal bundle, where $\bar{M}$ is a pseudo-Riemannian manifold with metric $\bar{g}$, and the fibers are non-degenerate with respect to $\bar{g}$. Suppose that H acts on $\bar{M}$ by isometries. Although it is not essential, throughout this chapter the action of isometries is understood as action on the left, and hence π is a left principal bundle. Let $\bar{x} \in \bar{M}$ and let $V_{\bar{x}}\bar{M}$ denote the vertical subspace at $\bar{x}$. If we take the orthogonal complement $H_{\bar{x}}\bar{M} = (V_{\bar{x}}\bar{M})^{\perp}$ of $V_{\bar{x}}\bar{M}$ in $T_{\bar{x}}\bar{M}$ with respect to the metric $\bar{g}$ we have

$$T_{\bar{x}}\bar{M} = V_{\bar{x}}\bar{M} \oplus H_{\bar{x}}\bar{M}. \tag{7.1}$$

Moreover, as H acts by isometries, the horizontal subspaces $H_{\bar{x}}\bar{M}$ are preserved by the action of H, and (7.1) leads to a connection in the principal bundle $\bar{M} \to M$ sometimes called *mechanical connection* (see for instance [43, 47]). In this situation there is a unique Riemannian metric g in M such that the restriction $\pi_* : H_{\bar{x}}\bar{M} \to T_{\pi(\bar{x})}M$ is an isometry at every $\bar{x} \in \bar{M}$. Obviously, the metric g satisfies

$$g(X, Y) \circ \pi = \bar{g}(X^H, Y^H) \qquad \forall X, Y \in \mathfrak{X}(M) \tag{7.2}$$

where X^H and Y^H denote the horizontal lift of X and Y with respect to the mechanical connection. To complete the notation, in the following we will denote by $Z^h \in H_{\bar{x}}\bar{M}$ the horizontal part of $Z \in T_{\bar{x}}\bar{M}$ with respect to the mechanical connection.

Proposition 7.0.4 *In the situation above, if $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$, then the Levi-Civita connection ∇ of the reduced metric g is given by*

$$\nabla_X Y = \pi_*(\bar{\nabla}_{X^H} Y^H), \qquad \forall X, Y \in \mathfrak{X}(M). \tag{7.3}$$

Proof. Since the structure group H acts by isometries, it also acts by affine transformations of $\bar{\nabla}$. The vector field $\bar{\nabla}_{X^H} Y^H$ is thus projectable and the operator $D_X Y = \pi_*(\bar{\nabla}_{X^H} Y^H)$ is well defined. As a direct computation shows, D fulfills the properties of a linear connection on M. From (7.2), for $X, Y, Z \in \mathfrak{X}(M)$ we have

$$\begin{aligned} g(D_X Y, Z) \circ \pi + g(Y, D_X, Z) \circ \pi &= \bar{g}((\bar{\nabla}_{X^H} Y^H)^h, Z^H) + \bar{g}(Y^H, (\bar{\nabla}_{X^H} Z^H)^h) \\ &= \bar{g}(\bar{\nabla}_{X^H} Y^H, Z^H) + \bar{g}(Y^H, \bar{\nabla}_{X^H} Z^H) \\ &= X^H(\bar{g}(Y^H, Z^H)). \end{aligned}$$

Hence $g(D_X Y, Z) + g(Y, D_X Z) = X(g(Y, Z))$, so that the connection D is metric. Finally, as $[X, Y]^H = [X^H, Y^H]^h$, the torsion tensor of D is

$$\begin{aligned} T(X, Y) &= D_X Y - D_Y X - [X, Y] \\ &= \pi_*(\bar{\nabla}_{X^H} Y^H - \bar{\nabla}_{Y^H} X^H - [X^H, Y^H]) \\ &= 0, \end{aligned}$$

whence D is the Levi-Civita connection of g. ■

Suppose now that $(\bar{M}, \bar{g})$ admits a homogeneous pseudo-Riemannian structure $\bar{S}$. We shall study under which conditions $\bar{S}$ induces a homogeneous pseudo-Riemannian structure $\bar{S}$ on the reduced manifold (M, g).

7.1 Reduction by a normal subgroup of isometries

As a first approach to the problem, we study a reduction procedure in a special but very interesting scenario, namely, when $(\bar{M}, \bar{g})$ is reduced by the action of a normal subgroup of isometries.

Let $(\bar{M}, \bar{g})$ be a homogeneous pseudo-Riemannian manifold. Let $\bar{G}$ be a group of isometries acting transitively on $\bar{M}$ and $H \triangleleft \bar{G}$ a normal subgroup acting freely on $\bar{M}$. The quotient $M = \bar{M}/H$ is thus endowed with a smooth structure such that $\pi : \bar{M} \to M$ is an H-principal bundle. If the fibers are non-degenerate, the bundle $\pi : \bar{M} \to M$ is equipped with the mechanical connection and M is pseudo-Riemannian with the reduced metric g as in (7.2). Since H is normal, there is a well-defined action of the group $G = \bar{G}/H$ on M given by

$$\begin{array}{rccl} L : & G \times M & \to & M \\ & ([\bar{a}], [\bar{x}]) & \mapsto & L_{[\bar{a}]}([\bar{x}]) = [L_{\bar{a}}(\bar{x})] \end{array} \tag{7.4}$$

where $[\bar{a}]$ and $[\bar{x}]$ denote the classes modulo H of $\bar{a} \in \bar{G}$ and $\bar{x} \in \bar{M}$ respectively, and $L_{\bar{a}}$ denotes the action of $\bar{G}$ on $\bar{M}$. The action of G is obviously transitive but needs not be effective. If it is not, we replace G by G/N, where N is the kernel of the map $G \to \mathit{Isom}(M)$, $a \mapsto L_a$, $a \in G$.

Proposition 7.1.1 *The group G acts on (M, g) by isometries.*

Proof. The action (7.4) can be written as $\pi \circ L_{\bar{a}} = L_a \circ \pi$, for $a = [\bar{a}]$. This implies that $\bar{G}$ preserves vertical subspaces and, acting by isometries, also their horizontal complements. Hence, the horizontal lift of $(L_a)_*(X)$ is $(L_{\bar{a}})_*(X^H)$ for all $X \in \mathfrak{X}(M)$. In addition, for $X, Y \in \mathfrak{X}(M)$

$$\begin{aligned} g\left((L_a)_*(X), (L_a)_*(Y)\right) \circ \pi &= \bar{g}\left((L_a)_*(X)^H, (L_a)_*(Y)^H\right) \\ &= \bar{g}\left((L_{\bar{a}})_*(X^H), (L_{\bar{a}})_*(Y^H)\right) \\ &= \bar{g}\left(X^H, Y^H\right) \\ &= g\left(X, Y\right) \circ \pi \end{aligned}$$

,and then L_a is an isometry. ■

From this last Proposition, the manifold (M, g) is homogeneous pseudo-Riemannian. We will call it the *reduced homogeneous pseudo-Riemannian manifold.*

Remark 7.1.2 *Note that Proposition 7.1.1 shows that the horizontal distribution is invariant by $\bar{G}$. This means that the mechanical connection is $\bar{G}$-invariant, an important fact that will be used in §7.2.*

Let $\bar{x} \in \bar{M}$ and $x = \pi(\bar{x}) \in M$. We denote by $\bar{K}$ the isotropy group of $\bar{x}$ under the action of $\bar{G}$, and by K the corresponding isotropy group of x under the action of G. We also denote their Lie algebras by $\bar{\mathfrak{k}}$ and $\mathfrak{k}$ respectively.

Lemma 7.1.3 *Let $\tau : \bar{G} \to G$ be the quotient homomorphism. Then $K = \tau(\bar{K})$ and the restriction $\tau|_{\bar{K}} : \bar{K} \to K$ is an isomorphism of Lie groups.*

Proof. It is obvious from (7.4) that $\tau(\bar{K}) \subset K$. Let now $k \in K$ and take $\bar{a} \in \bar{G}$ such that $k = \tau(\bar{a})$. For any $x \in M$, we have $x = L_k(x) = \pi(L_{\bar{a}}(\bar{x}))$, and then $L_{\bar{a}}(\bar{x})$ is in the same fiber as $\bar{x}$. Hence there exists $h \in H$ such that $L_h \circ L_{\bar{a}}(\bar{x}) = \bar{x}$, so that $h\bar{a} \in \bar{K}$. Since $\tau(h\bar{a}) = \tau(\bar{a}) = k$ we have $k \in \tau(\bar{K})$. For the injectivity of $\tau|_{\bar{K}}$, let $\bar{k}_1, \bar{k}_2 \in \bar{K}$ be such that $\tau(\bar{k}_1) = \tau(\bar{k}_2)$. There exists $h \in H$ such that $h\bar{k}_1 = \bar{k}_2$. Then $\bar{k}_1^{-1}h\bar{k}_1 = \bar{k}_1^{-1}\bar{k}_2$, so $\bar{k}_1^{-1}\bar{k}_2 \in \bar{K} \cap H$. But since H acts freely, $\bar{k}_1^{-1}\bar{k}_2 = \bar{e}$, and then $\bar{k}_1 = \bar{k}_2$. ■

Suppose now that $\bar{M} = \bar{G}/\bar{K}$ is reductive with reductive decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$. Let $\bar{\mu}$ be the infinitesimal action of $\bar{\mathfrak{g}}$ at the point $\bar{x}$, that is

$$\begin{array}{rccl} \bar{\mu}: & \bar{\mathfrak{g}} & \to & T_{\bar{x}}\bar{M} \\ & \bar{\xi} & \mapsto & \frac{d}{dt}\Big|_{t=0} L_{exp(t\bar{\xi})}(\bar{x}). \end{array}$$

Then for all $\bar{k} \in \bar{K}$ the following diagram is commutative

$$\begin{tikzcd} \bar{\mathfrak{g}} \arrow[r, "\mu"] \arrow[d, "\mathrm{Ad}(\bar{k})"'] \arrow[dr, phantom, "\circlearrowleft"] & T_{\bar{x}}\bar{M} \arrow[d, "(L_{\bar{k}})_*"] \\ \bar{\mathfrak{g}} \arrow[r, "\bar{\mu}"] & T_{\bar{x}}\bar{M} \end{tikzcd} \tag{7.5}$$

The restriction of $\bar{\mu}$ to $\bar{\mathfrak{m}}$ gives an isomorphism $\bar{\mu} : \bar{\mathfrak{m}} \to T_{\bar{x}}\bar{M}$, and the canonical connection $\tilde{\bar{\nabla}}$ with respect to the reductive decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ is determined by its value at $\bar{x}$

$$\left(\tilde{\bar{\nabla}}_{\bar{X}}\bar{Y}\right)_{\bar{x}} = \bar{\mu}\left([\bar{\mu}^{-1}(\bar{X}), \bar{\mu}^{-1}(\bar{Y})]_{\bar{\mathfrak{m}}}\right), \qquad \bar{X}, \bar{Y} \in T_{\bar{x}}\bar{M}. \tag{7.6}$$

Theorem 7.1.4 *Let $(\bar{M}, \bar{g})$ be a connected reductive homogeneous pseudo-Riemannian manifold, and let $\bar{G}$ be a group of isometries acting transitively and effectively on $\bar{M}$. Let $H \lhd \bar{G}$ be a normal subgroup acting freely on $\bar{M}$. Then, every homogeneous structure tensor $\bar{S}$ associated to $\bar{G}$ induces a homogeneous structure tensor S associated to $G = \bar{G}/H$ in the reduced pseudo-Riemannian manifold $M = \bar{M}/H$.*

Proof. Let $\bar{x} \in \bar{M}$ and $x = \pi(\bar{x}) \in M$, and let $\bar{\mathfrak{g}}$ be the Lie algebra of $\bar{G}$. For any reductive decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ associated to $\bar{S}$, the restricted isomorphism $\bar{\mu} : \bar{\mathfrak{m}} \to T_{\bar{x}}\bar{M}$ induces a decomposition

$$\bar{\mathfrak{m}} = \bar{\mathfrak{m}}^v \oplus \bar{\mathfrak{m}}^h$$

from $\bar{g}$ which is $\mathrm{Ad}(\bar{K})$-invariant by the commutativity of (7.5).

Let $\mathfrak{g} = \bar{\mathfrak{g}}/\mathfrak{h}$ be the Lie algebra of G and $\mu : \mathfrak{g} \to T_xM$ the corresponding infinitesimal action at x. For any $\bar{\xi} \in \bar{\mathfrak{g}}$, by (7.4) we have

$$\begin{aligned}
\pi_* \circ \bar{\mu}(\bar{\xi}) &= \pi_* \left(\left.\frac{d}{dt}\right|_{t=0} L_{\exp(t\bar{\xi})}(\bar{x}) \right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\pi \circ L_{\exp(t\bar{\xi})} \right)(\bar{x}) \\
&= \left.\frac{d}{dt}\right|_{t=0} L_{\tau(\exp(t\bar{\xi}))}(\pi(\bar{x})) \\
&= \left.\frac{d}{dt}\right|_{t=0} L_{\exp(t\tau_*(\bar{\xi}))}(x) \\
&= \mu \circ \tau_*(\bar{\xi}),
\end{aligned}$$

which means that the following diagram is commutative

$$\begin{tikzcd}
\bar{\mathfrak{g}} \arrow[r, "\bar{\mu}"] \arrow[d, "\tau_*"'] \arrow[dr, phantom, "\circlearrowleft"] & T_{\bar{x}}\bar{M} \arrow[d, "\pi_*"] \\
\mathfrak{g} \arrow[r, "\mu"] & T_xM.
\end{tikzcd} \tag{7.7}$$

Restrictions to $\bar{\mathfrak{m}}^h$ and $\bar{\mathfrak{m}}^v$ give commutative diagrams

$$\begin{tikzcd}
\bar{\mathfrak{m}}^v \arrow[r, "\bar{\mu}"] \arrow[d, "\tau_*"'] \arrow[dr, phantom, "\circlearrowleft"] & V_{\bar{x}}\bar{M} \arrow[d, "\pi_*"] \\
\tau_*(\bar{\mathfrak{m}}^v) \arrow[r, "\mu"] & \{0\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{\mathfrak{m}}^h \arrow[r, "\bar{\mu}"] \arrow[d, "\tau_*"'] \arrow[dr, phantom, "\circlearrowleft"] & H_{\bar{x}}\bar{M} \arrow[d, "\pi_*"] \\
\tau_*(\bar{\mathfrak{m}}^h) \arrow[r, "\mu"] & T_xM
\end{tikzcd} \tag{7.8}$$

showing that $\tau_* : \bar{\mathfrak{m}}^h \to \tau_*(\bar{\mathfrak{m}}^h)$ and $\mu : \tau_*(\bar{\mathfrak{m}}^h) \to T_xM$ are isomorphisms, and $\tau_*(\bar{\mathfrak{m}}^v) \subset \mathfrak{k}$. In addition, by Lemma 7.1.3 the restriction of $\tau_* : \bar{\mathfrak{g}} \to \mathfrak{g}$ to $\bar{\mathfrak{k}}$ is an isomorphism of Lie algebras from $\bar{\mathfrak{k}}$ to $\mathfrak{k}$. Therefore, denoting by $\mathfrak{m}$ the image $\tau_*(\bar{\mathfrak{m}}^h)$, we have

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}. \tag{7.9}$$

Let $k \in K$ and $\xi \in \mathfrak{m}$, and let $\bar{k} \in \bar{K}$ and $\bar{\xi} \in \bar{\mathfrak{m}}^h$ be such that $\tau(\bar{k}) = k$ and $\tau_*(\bar{\xi}) = \xi$ we have

$$\begin{aligned}
\mathrm{Ad}_k(\xi) &= \mathrm{Ad}_{\tau(\bar{k})}(\tau_*(\bar{\xi})) \\
&= \mu^{-1} \circ L_{\tau(\bar{k})} \circ \mu(\tau_*(\bar{\xi})) \\
&= \mu^{-1} \circ L_{\tau(\bar{k})} \circ \pi_*(\bar{\mu}(\bar{\xi})) \\
&= \mu^{-1} \circ \pi_* \circ L_{\bar{k}}(\bar{\mu}(\bar{\xi})) \\
&= \mu^{-1} \circ \pi_* \circ \bar{\mu}(\mathrm{Ad}_{\bar{k}}(\bar{\xi})) \\
&= \mu^{-1} \circ \mu \circ \tau_*(\mathrm{Ad}_{\bar{k}}(\bar{\xi})) \\
&= \tau_* \left(\mathrm{Ad}_{\bar{k}}(\bar{\xi}) \right).
\end{aligned}$$

Since $\bar{\mathfrak{m}}^h$ is $\mathrm{Ad}(\bar{K})$-invariant we deduce that $\mathrm{Ad}_k(\mathfrak{m}) \subset \tau_*(\bar{\mathfrak{m}}^h) = \mathfrak{m}$, whence (7.9) is a reductive decomposition.

The homogeneous structure tensor associated to (7.9) at x is given by

$$(S_x)_X Y = (\nabla_Y \xi^*)_x \qquad X, Y \in T_xM$$

where ξ^* is the fundamental vector field associated to $\xi \in \mathfrak{m}$ with $\xi^*_x = \mu(\xi) = X$. Let $\bar{\xi} \in \bar{\mathfrak{m}}^h$ be such that $\tau_*(\bar{\xi}) = \xi$, then

$$\begin{aligned}
(S_x)_X Y = (\nabla_Y \xi^*)_x &= \pi_* \left((\bar{\nabla}_{Y^H} (\xi^*)^H)_{\bar{x}} \right) \\
&= \pi_* \left((\bar{\nabla}_{Y^H} \bar{\xi}^*) \right) - \pi_* \left((\bar{\nabla}_{Y^H} (\bar{\xi}^*)^v)_{\bar{x}} \right).
\end{aligned}$$

Let $\bar{Z} \in T_{\bar{x}}\bar{M}$ be an horizontal vector, since $\bar{\xi}^*_{\bar{x}}$ is horizontal

$$\bar{g}\left((\bar{\nabla}_{Y^H}(\bar{\xi}^*)^v)_{\bar{x}}, \bar{Z}\right) = Y^H \bar{g}\left((\bar{\xi}^*)^v, \bar{Z}\right) - \bar{g}\left((\bar{\xi}^*)^v_{\bar{x}}, \bar{\nabla}_{Y^H}\bar{Z}\right) = 0.$$

Hence by (7.8)

$$(S_x)_X Y = \pi_*\left((\bar{S}_{\bar{x}})_{X^H} Y^H\right) \qquad X, Y \in T_x M. \tag{7.10}$$

Finally we extend S_x to the whole M with the action of G to obtain a homogeneous structure tensor S. ■

We shall call the tensor field S the *reduced homogeneous structure tensor*.

Corollary 7.1.5 *The reduced homogeneous structure can be expressed as*

$$S_X Y = \pi_*\left(\bar{S}_{X^H} Y^H\right) \qquad X, Y \in \mathfrak{X}(M). \tag{7.11}$$

Proof. Let $\bar{a} \in \bar{G}$ and $a = \tau(\bar{a}) \in G$ we proved that the horizontal lift of $(L_a)_*(X)$ is $(L_{\bar{a}})_*(X^H)$ for all $X \in \mathfrak{X}(M)$. This together with the invariance of $\bar{S}$ by $\bar{G}$ and the invariance of S by G gives (7.11). ■

7.1.1 The space of tensors reducing to a given tensor

Suppose that we are in the situation of Theorem 7.1.4, and we have a homogeneous structure tensor S associated to G in the reduced manifold M. Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ be a reductive decomposition associated to S, making use of (7.7) we define subspaces of $\bar{\mathfrak{g}}$

$$\bar{\mathfrak{m}}^h = \tau_*^{-1}(\mathfrak{m}) \cap \bar{\mu}^{-1}(H_{\bar{x}}\bar{M}) \qquad \text{and} \qquad \bar{\mathfrak{m}}^v = \mathfrak{h}.$$

Therefore, the decomposition

$$\bar{\mathfrak{g}} = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}, \qquad \text{with} \qquad \bar{\mathfrak{m}} = \bar{\mathfrak{m}}^v \oplus \bar{\mathfrak{m}}^h \tag{7.12}$$

is reductive. Indeed, since H is normal in $\bar{G}$, it is obvious that $\mathrm{Ad}(\bar{K})(\mathfrak{h}) \subset \mathfrak{h}$. On the other hand, for $\bar{k} \in \bar{K}$ and $\bar{\xi} \in \bar{\mathfrak{m}}^h$, as $\bar{\mu}(\mathrm{Ad}_{\bar{k}}(\bar{\xi})) = (L_{\bar{k}})_*(\bar{\mu}(\bar{\xi}))$, we have $\bar{\mu}(\mathrm{Ad}_{\bar{k}}(\bar{\xi})) \in H_{\bar{x}}\bar{M}$ and $\tau_*\left(\mathrm{Ad}_{\bar{k}}(\bar{\xi})\right) \in \mathfrak{m}$, and then $\mathrm{Ad}_{\bar{k}}(\bar{\xi}) \in \bar{\mathfrak{m}}^h$. The homogeneous structure tensor associated to this decomposition at $\bar{x}$ is given by (see for instance [31])

$$2(\bar{S}_{\bar{x}})_{\bar{X}\bar{Y}\bar{Z}} = B([\bar{\xi}, \bar{\eta}]_{\bar{\mathfrak{m}}}, \bar{\zeta}) - B([\bar{\eta}, \bar{\zeta}]_{\bar{\mathfrak{m}}}, \bar{\xi}) + B([\bar{\zeta}, \bar{\xi}]_{\bar{\mathfrak{m}}}, \bar{\eta}), \qquad \bar{X}, \bar{Y}, \bar{Z} \in T_{\bar{x}}\bar{M}, \tag{7.13}$$

where $\bar{\xi}, \bar{\eta}, \bar{\zeta} \in \bar{\mathfrak{m}}$ are such that their images by $\bar{\mu}$ are X, Y, Z, and B is the bilinear form on $\bar{\mathfrak{m}}$ inherited from $\bar{g}_{\bar{x}}$. Note that we are exactly in the situation of the proof of Theorem 7.1.4, so that the homogeneous structure tensor $\bar{S}$ associated to (7.12) reduces to S.

We can construct all other homogeneous structures in $\bar{M}$ associated to $\bar{G}$ by changing $\bar{\mathfrak{m}}$ in (7.12) by the graph

$$\bar{\mathfrak{m}}^{\varphi} = \{X + \varphi(X) / \, X \in \bar{\mathfrak{m}}\}$$

of an $\mathrm{Ad}(\bar{K})$-equivariant map $\varphi : \mathfrak{h} \oplus \bar{\mathfrak{m}}^h \to \bar{\mathfrak{k}}$. The condition that the new homogeneous structure tensors reduces to S is equivalent to $\varphi_{|\bar{\mathfrak{m}}^h} = 0$. The family of homogeneous structure tensors that reduce to S is thus parameterized by the set of $\mathrm{Ad}(\bar{K})$-equivariant maps $\varphi : \mathfrak{h} \to \bar{\mathfrak{k}}$. For the sake of simplicity we will denote by φ both $\varphi : \mathfrak{h} \to \bar{\mathfrak{k}}$ and its extension by zero to $\bar{\mathfrak{m}} = \mathfrak{h} \oplus \bar{\mathfrak{m}}^h$. The expression of the homogeneous structure tensor $\bar{S}^{\varphi}$ associated to this map is the same as in (7.13) by changing $\bar{\mathfrak{m}}$ by $\bar{\mathfrak{m}}^{\varphi}$, B by the induced bilinear form B^{φ} in $\bar{\mathfrak{m}}^{\varphi}$, and $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ by $\bar{\xi}' = \bar{\xi} + \varphi(\bar{\xi})$, $\bar{\eta}' = \bar{\eta} + \varphi(\bar{\eta})$, $\bar{\zeta}' = \bar{\zeta} + \varphi(\bar{\zeta}) \in \bar{\mathfrak{m}}^{\varphi}$ respectively. As

$$[\bar{\xi}', \bar{\eta}']_{\bar{\mathfrak{m}}^{\varphi}} = [\bar{\xi}, \bar{\eta}]_{\bar{\mathfrak{m}}^{\varphi}} + [\bar{\xi}, \varphi(\bar{\eta})]_{\bar{\mathfrak{m}}^{\varphi}} + [\varphi(\bar{\xi}), \bar{\eta}]_{\bar{\mathfrak{m}}^{\varphi}} + [\varphi(\bar{\xi}), \varphi(\bar{\eta})]_{\bar{\mathfrak{m}}^{\varphi}}$$

and $[\varphi(\bar{\xi}), \varphi(\bar{\eta})]_{\bar{\mathfrak{m}}^\varphi} = 0$, we have that

$$\begin{aligned} B^\varphi\left([\bar{\xi}', \bar{\eta}']_{\bar{\mathfrak{m}}^\varphi}, \bar{\zeta}'\right) &= B^\varphi\left([\bar{\xi}, \bar{\eta}]_{\bar{\mathfrak{m}}^\varphi}, \bar{\zeta}'\right) + B^\varphi\left([\bar{\xi}, \varphi(\bar{\eta})]_{\bar{\mathfrak{m}}^\varphi} + [\varphi(\bar{\xi}), \bar{\eta}]_{\bar{\mathfrak{m}}^\varphi}, \bar{\zeta}'\right) \\ &= B\left([\bar{\xi}, \bar{\eta}]_{\bar{\mathfrak{m}}}, \bar{\zeta}\right) + B\left([\bar{\xi}, \varphi(\bar{\eta})] + [\varphi(\bar{\xi}), \bar{\eta}], \bar{\zeta}\right), \end{aligned}$$

where one has to take into account that the isomorphism $\bar{\mathfrak{m}} \to \bar{\mathfrak{m}}^\varphi$, $\bar{\xi} \mapsto \bar{\xi} + \varphi(\bar{\xi})$ is an isometry with respect to B and B^φ. Therefore

$$\begin{aligned} (\bar{S}^\varphi_{\bar{x}})_{\bar{X}\bar{Y}\bar{Z}} = (\bar{S}_{\bar{x}})_{\bar{X}\bar{Y}\bar{Z}} &+ \frac{1}{2}\left\{B\left([\bar{\xi}, \varphi(\bar{\eta})] + [\varphi(\bar{\xi}), \bar{\eta}], \bar{\zeta}\right)\right. \\ &- B\left([\bar{\eta}, \varphi(\bar{\zeta})] + [\varphi(\bar{\eta}), \bar{\zeta}], \bar{\xi}\right) + \left.B\left([\bar{\zeta}, \varphi(\bar{\xi})] + [\varphi(\bar{\zeta}), \bar{\xi}], \bar{\eta}\right)\right\}. \end{aligned} \tag{7.14}$$

The summands involving B define a tensor field P^φ globally defined in $\bar{M}$ by the left action of $\bar{G}$. More precisely, for any $\bar{y} \in \bar{M}$, with $\bar{y} = L_{\bar{a}}(\bar{x})$, $\bar{a} \in \bar{G}$, this tensor is

$$\begin{aligned} (P^\varphi_{\bar{y}})_{\bar{X}\bar{Y}\bar{Z}} &= \frac{1}{2}\left\{B_{\bar{y}}\left([\bar{\xi}, \varphi_{\bar{y}}(\bar{\eta})] + [\varphi_{\bar{y}}(\bar{\xi}), \bar{\eta}], \bar{\zeta}\right) - B_{\bar{y}}\left([\bar{\eta}, \varphi_{\bar{y}}(\bar{\zeta})] + [\varphi_{\bar{y}}(\bar{\eta}), \bar{\zeta}], \bar{\xi}\right)\right. \\ &+ \left.B_{\bar{y}}\left([\bar{\zeta}, \varphi_{\bar{y}}(\bar{\xi})] + [\varphi_{\bar{y}}(\bar{\zeta}), \bar{\xi}], \bar{\eta}\right)\right\}, \end{aligned} \tag{7.15}$$

for $\bar{X}, \bar{Y}, \bar{Z} \in T_{\bar{y}}\bar{M}$, where

$$\bar{\mathfrak{m}}_{\bar{y}} := \mathrm{Ad}(\bar{a})(\bar{\mathfrak{m}}), \qquad \bar{\mathfrak{k}}_{\bar{y}} := \mathrm{Ad}(\bar{a})(\bar{\mathfrak{k}}),$$

$$\varphi_{\bar{y}} := \mathrm{Ad}(\bar{a}) \circ \varphi \circ \mathrm{Ad}(\bar{a}^{-1}) : \mathfrak{h} \to \bar{\mathfrak{k}}_{\bar{y}},$$

$B_{\bar{y}}$ is the bilinear form on $\bar{\mathfrak{m}}_{\bar{y}}$ induced from $\bar{g}_{\bar{y}}$ by

$$\bar{\mu}_{\bar{y}} := (L_{\bar{a}})_* \circ \bar{\mu} \circ \mathrm{Ad}(\bar{a}^{-1}) : \bar{\mathfrak{m}}_{\bar{y}} \to T_{\bar{y}}\bar{M},$$

and $\bar{\xi}, \bar{\eta}, \bar{\zeta} \in \bar{\mathfrak{m}}_{\bar{y}}$ are such that their images by $\bar{\mu}_{\bar{y}}$ are $\bar{X}, \bar{Y}, \bar{Z}$ respectively. We have thus proved

Proposition 7.1.6 *In the situation of Theorem 7.1.4, let S be a homogeneous structure tensor in M associated to G. Then the space of homogeneous structure tensors in $\bar{M}$ associated to $\bar{G}$ and reducing to S is a vector space isomorphic to the space of* $\mathrm{Ad}(\bar{K})$*-equivariant maps $\varphi : \mathfrak{h} \to \bar{\mathfrak{k}}$. Moreover, the isomorphism is given by*

$$\varphi \mapsto \bar{S}^\varphi = \bar{S} + P^\varphi,$$

where $\bar{S}$ is the homogeneous structure associated to the decomposition (7.12) and P^φ is given in (7.15).

7.2 Reduction in a principal bundle

In this section we show a reduction procedure in a more general framework. But in the first place, in order to find suitable conditions under which this reduction procedure is possible, we look again at the case studied in the previous section. More precisely, recall (see Remark 7.1.2) that the normality of the group H gave the invariance of the mechanical connection. This implies that the connection form of the mechanical connection ω is $\mathrm{Ad}(\bar{G})$-equivariant, i.e.,

$$L^*_{\bar{a}}\omega = \mathrm{Ad}(\bar{a}) \cdot \omega, \qquad \bar{a} \in \bar{G}, \tag{7.16}$$

where $\mathrm{Ad}_{\bar{a}} \cdot \omega$ denotes the 1-form in $\bar{M}$ with values in $\mathfrak{h}$ given by

$$(\mathrm{Ad}_{\bar{a}} \cdot \omega)(\bar{X}) = \mathrm{Ad}_{\bar{a}}(\omega(\bar{X})).$$

Recall (Proposition 1.3.12) that the canonical connection $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}$ associated to the reductive decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ at $\bar{x}$ is characterized by the following property: for every $\bar{\xi} \in \bar{\mathfrak{m}}$, the parallel displacement with respect to $\tilde{\bar{\nabla}}$ along the curve $\gamma(t) = L_{\exp(t\bar{\xi})}(\bar{x})$, from $\bar{x}$ to $\gamma(t)$, is equal to $(L_{\exp(t\bar{\xi})})_*$. This implies

$$\left(\tilde{\bar{\nabla}}_{\bar{X}}\omega\right)_{\bar{x}} = \mathrm{ad}_{\bar{\mu}^{-1}(\bar{X})} \cdot \omega_{\bar{x}}, \qquad \bar{X} \in T_{\bar{x}}\bar{M},$$

and by the invariance of $\tilde{\bar{\nabla}}$ by $\bar{G}$

$$\left(\tilde{\bar{\nabla}}_{\bar{X}}\omega\right)_{\bar{y}} = \mathrm{ad}_{\bar{\mu}_{\bar{y}}^{-1}(\bar{X})} \cdot \omega_{\bar{y}}, \qquad \bar{y} \in \bar{M}, \bar{X} \in T_{\bar{y}}\bar{M}, \tag{7.17}$$

that is, the covariant derivative of ω by the connection $\tilde{\bar{\nabla}}$ is proportional to itself by a suitable linear operator. We note that, in particular, if H is contained in the center of $\bar{G}$, the linear operator is null, hence ω is invariant by $\bar{G}$. If H is just a normal subgroup not contained in the center, condition (7.17) comes from the equivariance of ω.

We now turn to the general case, namely, we begin with any homogeneous pseudo-Riemannian structure $\bar{S}$ on a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$ where a group H acts by isometries, and such that $\bar{M} \to \bar{M}/H = M$ is a principal bundle with non-degenerate fibers. The preceding discussion suggests that for the reduction procedure to be possible we may need to add an algebraic condition for the mechanical connection analogous to (7.17). This suggestion is correct as we can see in the following result.

Theorem 7.2.1 *Let $(\bar{M}, \bar{g})$ be a pseudo-Riemannian manifold. Let $\pi : \bar{M} \to M$ be a principal bundle with non-degenerate fibers, and with structure group H acting on $\bar{M}$ by isometries. Let ω be the connection 1-form of the mechanical connection. Let $\bar{S}$ be an H-invariant homogeneous pseudo-Riemannian structure with associated AS-connection $\tilde{\bar{\nabla}}$ such that*

$$\tilde{\bar{\nabla}}\omega = \alpha \cdot \omega \tag{7.18}$$

for some 1-form α on $\bar{M}$ taking values in $\mathrm{End}(\mathfrak{h})$. *Then, the tensor field S defined by*

$$S_X Y = \pi_*\left(\bar{S}_{X^H} Y^H\right) \qquad X, Y \in \mathfrak{X}(M) \tag{7.19}$$

is a homogeneous pseudo-Riemannian structure on (M, g), where g is the reduced Riemannian metric.

Proof. We note in the first place that the H-invariance of $\bar{S}$ implies that $\bar{S}_{X^H}Y^H$ is projectable so that then S is well defined. Since the structure group H acts by isometries, the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$ is H-invariant, which implies that $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}$ is also H-invariant. From (7.18) we have that for all $X, Y \in \mathfrak{X}(M)$

$$\omega(\tilde{\bar{\nabla}}_{X^H}Y^H) = X^H\left(\omega(Y^H)\right) - \left(\tilde{\bar{\nabla}}_{X^H}\omega\right)(Y^H) = -\alpha(X^H) \cdot \omega(Y^H) = 0,$$

so that $\tilde{\bar{\nabla}}_{X^H}Y^H$ is horizontal. If we define $\tilde{\nabla} = \nabla - S$, ∇ being the Levi-Civita connection of g, then $\tilde{\bar{\nabla}}_{X^H}Y^H$ projects to $\tilde{\nabla}_{X^H}Y^H$. Hence by H-invariance,

$$\left(\tilde{\nabla}_X Y\right)^H = \tilde{\bar{\nabla}}_{X^H}Y^H. \tag{7.20}$$

We now prove that S satisfies

$$\tilde{\nabla} g = 0, \qquad \tilde{\nabla}\tilde{R} = 0, \qquad \tilde{\nabla} S = 0, \tag{7.21}$$

where $\tilde{R}$ is the curvature tensor of $\tilde{\nabla}$, and $\tilde{R}$ and S are seen as tensor fields of type $(0,4)$ and $(0,3)$ respectively. Recall that those equations are equivalent to Ambrose-Singer equations (Proposition 2.2.2).

For the first equation, taking into account (7.20), we have

$$\begin{aligned}\left(\widetilde{\nabla}_U g\right)(X,Y)\circ\pi &= U\left(g(X,Y)\right)\circ\pi - g(\widetilde{\nabla}_U X,Y)\circ\pi - g(X,\widetilde{\nabla}_U Y)\circ\pi\\ &= U^H\left(\bar g(X^H,Y^H)\right) - \bar g\left((\widetilde{\nabla}_U X)^H,Y^H\right) - \bar g\left(X^H,(\widetilde{\nabla}_U Y)^H\right)\\ &= U^H\left(\bar g(X^H,Y^H)\right) - \bar g\left(\widetilde{\bar\nabla}_{U^H}X^H,Y^H\right) - \bar g\left(X^H,\widetilde{\bar\nabla}_{U^H}Y^H\right)\\ &= \left(\widetilde{\bar\nabla}_{U^H}\bar g\right)(X^H,Y^H)\end{aligned}$$

for $U,X,Y\in\mathfrak{X}(M)$, whence $\widetilde{\nabla}g=0$ since $\widetilde{\bar\nabla}\bar g=0$. For the third equation, let $U,X,Y,Z\in\mathfrak{X}(M)$. Again by (7.20) we have

$$\begin{aligned}\left(\widetilde{\nabla}_U S\right)_{XYZ}\circ\pi &= U\left(S_{XYZ}\right)\circ\pi - \left(S_{\widetilde{\nabla}_U XYZ}\right)\circ\pi\\ &\quad - \left(S_{X\widetilde{\nabla}_U YZ}\right)\circ\pi - \left(S_{XY\widetilde{\nabla}_U Z}\right)\circ\pi\\ &= U^H\left(\bar S_{X^HY^HZ^H}\right) - \bar S_{(\widetilde{\nabla}_U X)^HY^HZ^H}\\ &\quad -\bar S_{X^H(\widetilde{\nabla}_U Y)^HZ^H} - \bar S_{X^HY^H(\widetilde{\nabla}_U Z)^H}\\ &= U^H\left(\bar S_{X^HY^HZ^H}\right) - \bar S_{\widetilde{\bar\nabla}_{U^H}X^HY^HZ^H}\\ &\quad -\bar S_{X^H\widetilde{\bar\nabla}_{U^H}Y^HZ^H} - \bar S_{X^HY^H\widetilde{\bar\nabla}_{U^H}Z^H}\\ &= \left(\widetilde{\bar\nabla}_{U^H}\bar S\right)_{X^HY^HZ^H},\end{aligned}$$

which vanishes as $\widetilde{\bar\nabla}\bar S=0$. We now prove the second equation in (7.21). Let $\widetilde{\bar R}$ be the curvature tensor of $\widetilde{\bar\nabla}$. From (7.20), for $X,Y,Z\in\mathfrak{X}(M)$ we first have

$$\begin{aligned}(\widetilde{R}_{XY}Z)^H &= \widetilde{\bar\nabla}_{X^H}(\widetilde{\nabla}_Y Z)^H - \widetilde{\bar\nabla}_{Y^H}(\widetilde{\nabla}_X Z)^H - \widetilde{\bar\nabla}_{[X,Y]^H}Z^H\\ &= \widetilde{\bar\nabla}_{X^H}(\widetilde{\bar\nabla}_{Y^H}Z^H) - \widetilde{\bar\nabla}_{Y^H}(\widetilde{\bar\nabla}_{X^H}Z^H) - \widetilde{\bar\nabla}_{[X^H,Y^H]^h}Z^H\\ &= \widetilde{\bar\nabla}_{X^H}(\widetilde{\bar\nabla}_{Y^H}Z^H) - \widetilde{\bar\nabla}_{Y^H}(\widetilde{\bar\nabla}_{X^H}Z^H) - \widetilde{\bar\nabla}_{[X^H,Y^H]}Z^H + \widetilde{\bar\nabla}_{[X^H,Y^H]^v}Z^H\\ &= \widetilde{\bar R}_{X^HY^H}Z^H + \widetilde{\bar\nabla}_{[X^H,Y^H]^v}Z^H.\end{aligned}$$

For $X,Y,Z,W\in\mathfrak{X}(M)$ one has

$$\begin{aligned}\widetilde{R}_{XYZW}\circ\pi &= \widetilde{\bar R}_{X^HY^HZ^HW^H} + \bar g\left(\widetilde{\bar\nabla}_{[X^H,Y^H]^v}Z^H,W^H\right)\\ &= \widetilde{\bar R}_{X^HY^HZ^HW^H} - \bar g\left(\widetilde{\bar\nabla}_{\Omega(X^H,Y^H)^*}Z^H,W^H\right),\end{aligned}\tag{7.22}$$

where $\Omega(X^H,Y^H)^*$ is the fundamental vector field associated to $\Omega(X^H,Y^H)\in\mathfrak{h}$. For any $\bar x\in\bar M$, let $\mathbb{I}(\bar x)$ be the non-degenerate bilinear form in $\mathfrak{h}$ defined as

$$\mathbb{I}(\bar x)(\xi,\eta)=\bar g(\xi^*_{\bar x},\eta^*_{\bar x}),\qquad \xi,\eta\in\mathfrak{h}.$$

Applying Koszul's formula for $\bar\nabla$ and taking into account that $[X^H,\xi^*]=0$ for any $X\in\mathfrak{X}(M)$ and $\xi\in\mathfrak{h}$, we have

$$\begin{aligned}\bar g\left(\widetilde{\bar\nabla}_{\Omega(X^H,Y^H)^*}Z^H,W^H\right) &= \bar g\left(\bar\nabla_{\Omega(X^H,Y^H)^*}Z^H,W^H\right) - \bar g\left(\bar S_{\Omega(X^H,Y^H)^*}Z^HW^H\right)\\ &= \frac{1}{2}\mathbb{I}\left(\Omega(X^H,Y^H),\Omega(Z^H,W^H)\right) - \bar S_{\Omega(X^H,Y^H)^*Z^HW^H},\end{aligned}$$

where, as usual, $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}$. Applying the previous equation and (7.22), a direct computation shows that

$$\begin{aligned}
\left(\widetilde{\nabla}_U \widetilde{R}\right)_{XYZW} \circ \pi = & \left(\tilde{\bar{\nabla}}_{U^H} \tilde{\bar{R}}\right)_{X^H Y^H Z^H W^H} \\
& - \frac{1}{2} U^H \left(\mathbb{I}(\Omega(X^H, Y^H), \Omega(Z^H, W^H))\right) \\
& + \frac{1}{2} \mathbb{I}\left(\Omega(\tilde{\bar{\nabla}}_{U^H} X^H, Y^H), \Omega(Z^H, W^H)\right) \\
& + \frac{1}{2} \mathbb{I}\left(\Omega(X^H, \tilde{\bar{\nabla}}_{U^H} Y^H), \Omega(Z^H, W^H)\right) \\
& + \frac{1}{2} \mathbb{I}\left(\Omega(X^H, Y^H), \Omega(\tilde{\bar{\nabla}}_{U^H} Z^H, W^H)\right) \\
& + \frac{1}{2} \mathbb{I}\left(\Omega(X^H, Y^H), \Omega(Z^H, \tilde{\bar{\nabla}}_{U^H} W^H)\right) \\
& + U^H \left(\bar{S}_{\Omega(X^H, Y^H)^* Z^H W^H}\right) - \bar{S}_{\Omega(\tilde{\bar{\nabla}}_{U^H} X^H, Y^H)^* Z^H W^H} \\
& - \bar{S}_{\Omega(X^H, \tilde{\bar{\nabla}}_{U^H} Y^H)^* Z^H W^H} - \bar{S}_{\Omega(X^H, Y^H)^* (\tilde{\bar{\nabla}}_{U^H} Z^H) W^H} \\
& - \bar{S}_{\Omega(X^H, Y^H)^* Z^H (\tilde{\bar{\nabla}}_{U^H} W^H)}.
\end{aligned} \tag{7.23}$$

On the other hand, by (7.18)

$$0 = \left(\tilde{\bar{\nabla}}_{X^H} \omega\right)(Y^H) - \left(\tilde{\bar{\nabla}}_{Y^H} \omega\right)(X^H) = d\omega(X^H, Y^H) - \omega\left(\tilde{\bar{T}}_{X^H} Y^H\right),$$

where $\tilde{\bar{T}}$ is the torsion tensor field of $\tilde{\bar{\nabla}}$. Since $\Omega(\bar{X}, \bar{Y}) = d\omega(\bar{X}^h, \bar{Y}^h)$ by definition, we have

$$\Omega(X^H, Y^H) = \omega\left(\tilde{\bar{T}}_{X^H} Y^H\right).$$

Making use of

$$\tilde{\bar{T}}_{X^H} Y^H = \bar{S}_{Y^H} X^H - \bar{S}_{X^H} Y^H,$$

together with (7.18) and $\tilde{\bar{\nabla}} \bar{S} = 0$, one has that

$$\left(\tilde{\bar{\nabla}}_{U^H} \Omega\right)(X^H, Y^H) = \alpha(U^H) \cdot \Omega(X^H, Y^H). \tag{7.24}$$

Now, from $\omega([X^H, Y^H]^v) = -\Omega(X^H, Y^H)$ and (7.18) we get

$$\omega\left(\tilde{\bar{\nabla}}_{U^H} [X^H, Y^H]^v\right) = -U^H\left(\Omega(X^H, Y^H)\right) + \alpha(U^H) \cdot \Omega(X^H, Y^H), \tag{7.25}$$

so that

$$\begin{aligned}
U^H\left(\mathbb{I}\left(\Omega(X^H, Y^H), \Omega(Z^H, W^H)\right)\right) = & \bar{g}\left(\tilde{\bar{\nabla}}_{U^H} [X^H, Y^H]^v, [Z^H, W^H]^v\right) \\
& + \bar{g}\left([X^H, Y^H]^v, \tilde{\bar{\nabla}}_{U^H} [Z^H, W^H]^v\right) \\
= & \, \mathbb{I}\left(U^H \Omega(X^H, Y^H), \Omega(Z^H, W^H)\right) \\
& - \mathbb{I}\left(\alpha(U^H) \cdot \Omega(X^H, Y^H), \Omega(Z^H, W^H)\right) \\
& + \mathbb{I}\left(\Omega(X^H, Y^H), U^H \Omega(Z^H, W^H)\right) \\
& - \mathbb{I}\left(\Omega(X^H, Y^H), \alpha(U^H) \cdot \Omega(Z^H, W^H)\right).
\end{aligned}$$

In addition, by (7.24) and (7.25)

$$\Omega(\tilde{\bar{\nabla}}_{U^H} X^H, Y^H) + \Omega(X^H, \tilde{\bar{\nabla}}_{U^H} Y^H) = -\omega\left(\tilde{\bar{\nabla}}_{U^H} [X^H, Y^H]^v\right),$$

whence

$$\Omega(\tilde{\bar{\nabla}}_{U^H}X^H, Y^H)^* + \Omega(X^H, \tilde{\bar{\nabla}}_{U^H}Y^H)^* = \tilde{\bar{\nabla}}_{U^H}\Omega(X^H, Y^H)^* \tag{7.26}$$

since $\tilde{\bar{\nabla}}_{U^H}[X^H, Y^H]^v$ is vertical. Substituting the preceding formulas and grouping terms, (7.23) becomes

$$\begin{aligned}
\left(\widetilde{\nabla}_U \widetilde{R}\right)_{XYZW} \circ \pi &= \left(\tilde{\bar{\nabla}}_{U^H} \tilde{\bar{R}}\right)_{X^H Y^H Z^H W^H} \\
&+ \frac{1}{2}\mathbb{I}\left((\tilde{\bar{\nabla}}_{U^H}\Omega)(X^H, Y^H), \Omega(Z^H, W^H)\right) \\
&- \frac{1}{2}\mathbb{I}\left(\alpha(U^H)\cdot\Omega(X^H, Y^H), \Omega(Z^H, W^H)\right) \\
&+ \frac{1}{2}\mathbb{I}\left(\Omega(X^H, Y^H), (\tilde{\bar{\nabla}}_{U^H}\Omega)(Z^H, W^H)\right) \\
&- \frac{1}{2}\mathbb{I}\left(\Omega(X^H, Y^H), \alpha(U^H)\cdot\Omega(Z^H, W^H)\right) \\
&- \left(\tilde{\bar{\nabla}}_{U^H}\bar{S}\right)_{\Omega(X^H, Y^H)^* Z^H W^H}.
\end{aligned}$$

Taking into account (7.24) and (7.26), we deduce that $\widetilde{\nabla}_U \widetilde{R} = 0$. This finishes the proof of Theorem 7.2.1. ■

Remark 7.2.2 *In the situation of Theorem 7.2.1, in the case $\bar{S}$ is a homogeneous structure tensor associated to a Lie group $\bar{G}$ acting by isometries in $\bar{M}$, one could ask if H can be seen as a normal subgroup of $\bar{G}$ and if the projected tensor S is associated to the group $G = \bar{G}/H$. The answer is not necessarily affirmative. More precisely, for a connected, simply connected and complete manifold $\bar{M}$, if we construct the group $\bar{G}$ from $\bar{S}$ following the proof of Ambrose-Singer Theorem (see §2.2), one can see that the normality of H is not guaranteed and the group $\bar{G}$ needs not project to the group G constructed in M from S by the same method. An example of this situation will be shown in subsection 7.2.1 (Hopf fibration case $\lambda = 0$).*

In the situation of Theorem 7.2.1, it is an interesting question if the classification obtained in §4.2.1 is respected by the reduction procedure.

Proposition 7.2.3 *The classes $\{0\}$, $\mathcal{S}_1$, $\mathcal{S}_3$, $\mathcal{S}_1 \oplus \mathcal{S}_2$ and $\mathcal{S}_1 \oplus \mathcal{S}_3$ are invariant under the reduction procedure.*

Proof. By the expression of the reduced structure tensor (7.19) it is obvious that if $\bar{S} = 0$ then $S = 0$. Let $\bar{S}$ be a tensor field in the class $\mathcal{S}_1$ given a vector field $\bar{\xi}$. Recall that $\bar{\xi}$ is parallel with respect to $\tilde{\bar{\nabla}}$. Since $\bar{S}$ is H-invariant the vector field $\bar{\xi}$ is also H-invariant, and then projectable. Let ξ be the projection of $\bar{\xi}$, we have $\xi^H = \bar{\xi}^h$ and then

$$\begin{aligned}
S_{XYZ} \circ \pi &= \bar{g}(X^H, Y^H)\bar{g}(\bar{\xi}, Z^H) - \bar{g}(Y^H, \bar{\xi})\bar{g}(X^H, Z^H) \\
&= \bar{g}(X^H, Y^H)\bar{g}(\xi^H, Z^H) - \bar{g}(Y^H, \xi^H)\bar{g}(X^H, Z^H) \\
&= g(X, Y)g(\xi, Z)\circ\pi - g(Y, \xi)g(X, Z)\circ\pi,
\end{aligned}$$

whence $S \in \mathcal{S}_1$. With a similar argument one proves that the class $\mathcal{S}_1 \oplus \mathcal{S}_2$ is also invariant. Regarding the classes $\mathcal{S}_3$ and $\mathcal{S}_1 \oplus \mathcal{S}_3$, they are characterized by algebraic conditions which are clearly preserved by the reduction formula (7.19). ■

The remaining classes $\mathcal{S}_2$ and $\mathcal{S}_2 \oplus \mathcal{S}_3$ are characterized by the vanishing of the trace c_{12}. Let $x \in M$ and $\{e_i\}_{i=1,\ldots,n}$ be an orthonormal basis of T_xM. For $X \in T_xM$

$$c_{12}(S)(X) = \sum_i \varepsilon^i S_{e_i e_i X} = \sum_i \varepsilon^i \bar{S}_{e_i^H e_i^H X^H} = c_{12}(\bar{S})(X^H) - \sum_j \varepsilon^j \bar{S}_{V_j V_j X^H}, \tag{7.27}$$

where $\{V_j\}_{j=1,\ldots,r}$ is an orthonormal basis of the vertical subspace $V_{\bar{x}}\bar{M}$, $\bar{x} \in \pi^{-1}(x)$. From $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}$ one has

$$\bar{S}_{V_j V_j X^H} = \bar{g}(\bar{\nabla}_{V_j} V_j, X^H) - \bar{g}(\tilde{\bar{\nabla}}_{V_j} V_j, X^H) = -\bar{g}(\bar{\nabla}_{V_j} X^H, V_j) + \bar{g}(\tilde{\bar{\nabla}}_{V_j} X^H, V_j),$$

where the vectors V_j, $j = 1, ..., r$, are extended to unitary and respectively orthogonal vertical vector fields. As from (7.18) we have

$$\omega(\tilde{\bar{\nabla}}_{V_j} X^H) = V_j(\omega(X^H)) - \alpha(V_j) \cdot \omega(X^H) = 0,$$

the second summand in the formula for $\bar{S}_{V_j V_j X^H}$ is zero, and then

$$\bar{S}_{V_j V_j X^H} = -\bar{g}(\bar{\nabla}_{V_j} X^H, V_j) = \bar{g}(B(V_j, V_j), X^H),$$

where B denotes the second fundamental form of the fiber $\pi^{-1}(x)$ at $\bar{x}$. Inserting this in (7.27) we obtain that

$$c_{12}(S)(X) = c_{12}(\bar{S})(X^H) - \sum_j \varepsilon^j \bar{g}(B(V_j, V_j), X^H) = c_{12}(\bar{S})(X^H) - \bar{g}(\mathrm{H}, X^H)$$

where H denotes the mean curvature operator (trace of B) of the fiber at $\bar{x}$. We have proved the following.

Proposition 7.2.4 *The classes $\mathcal{S}_2$ and $\mathcal{S}_2 \oplus \mathcal{S}_3$ are invariant under reduction if and only if the fibers of the principal bundle $\pi : (\bar{M}, \bar{g}) \to (M, g)$ are minimal Riemannian sub-manifolds of $(\bar{M}, \bar{g})$.*

Remark 7.2.5 *Propositions 7.2.3 and 7.2.4 (when the fibers are minimal) do not exclude that a homogeneous structure $\bar{S}$ in a class $\mathcal{S}_i \oplus \mathcal{S}_j$ reduces to a tensor S belonging to a smaller class $\mathcal{S}_i$ or $\mathcal{S}_j$, or even to the null tensor. We shall show some examples of these situations in the next section.*

7.2.1 Examples

(a) The fibration $\mathbb{R}H(n) \to \mathbb{R}H(n-1)$

The real n-dimensional hyperbolic space $(\mathbb{R}H(n), \bar{g})$

$$\mathbb{R}H(n) = \{(\bar{y}^0, \bar{y}^1, \ldots, \bar{y}^{n-1}) \in \mathbb{R}^n / \bar{y}^0 > 0\}$$

$$\bar{g} = \frac{1}{(\bar{y}^0)^2} \sum_{j=0}^{n-1} d\bar{y}^j \otimes d\bar{y}^j,$$

is a symmetric space, $\mathbb{R}H(n) = SO(n-1,1)/O(n-1)$. For the sake of simplicity we confine ourselves to the case $n = 4$. For general n the generalization is straightforward. Besides its symmetric description, all other groups of isometries acting transitively on $\mathbb{R}H(4)$ are of the type (see [17]) $\bar{G} = FN$, where F is a connected closed subgroup of $SO(3)A$ with nontrivial projection to A. In particular, we now consider

$$\bar{G} = SO(2)AN.$$

Geometrically, if we see $SO(2)$ as the isotropy group of the point $\bar{x} = (1, 0, 0, 0)$, its Lie algebra $\bar{\mathfrak{k}}$ consists of infinitesimal rotations generated by

$$r = \bar{y}^2 \frac{\partial}{\partial \bar{y}^3} - \bar{y}^3 \frac{\partial}{\partial \bar{y}^2}.$$

The subspace $\bar{\mathfrak{m}} = \mathfrak{a} \oplus \mathfrak{n}$, which is the Lie algebra of the factor AN, gives a reductive decomposition

$$\bar{\mathfrak{g}} = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}.$$

Let $a \in \mathfrak{a}$, $n_1, n_2, n_3 \in \mathfrak{n}$ be the generators of $\mathfrak{a}$ and $\mathfrak{n}$ respectively, where n_i is the infinitesimal translation in $\mathbb{R}H(4)$ in the direction of $\partial/\partial\bar{y}^i$. All other reductive decompositions $\bar{\mathfrak{g}} = \bar{\mathfrak{m}}^{\varphi} + \bar{\mathfrak{k}}$ associated to $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{k}}$ are given by the graph of any equivariant map $\varphi : \mathfrak{m} \to \mathfrak{k}$. As a computation shows, all these equivariant maps are

$$\begin{array}{rrcl} \varphi_{(\lambda_0,\lambda_1)}: & \mathfrak{m} & \to & \mathfrak{k} \\ & a & \mapsto & \lambda_0 r \\ & n_1 & \mapsto & \lambda_1 r \\ & n_2, n_3 & \mapsto & 0, \end{array}$$

with $\lambda_0, \lambda_1 \in \mathbb{R}$. The homogeneous structure tensors associated to this 2-parameter family of reductive decompositions are

$$\bar{S}^{(\lambda_0,\lambda_1)} = \frac{1}{(\bar{y}^0)^3}\left(\sum_{k=1}^{3} d\bar{y}^k \otimes d\bar{y}^k \wedge d\bar{y}^0 - \lambda_0 d\bar{y}^0 \otimes d\bar{y}^2 \wedge d\bar{y}^3 - \lambda_1 d\bar{y}^1 \otimes d\bar{y}^2 \wedge d\bar{y}^3\right),$$

and the canonical connection $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}^{(\lambda_0,\lambda_1)}$ (where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$) is thus given by

$$\begin{array}{c} \tilde{\bar{\nabla}}_{\partial_0}\partial_0 = -\frac{1}{\bar{y}^0}\partial_0, \quad \tilde{\bar{\nabla}}_{\partial_0}\partial_1 = -\frac{1}{\bar{y}^0}\partial_1, \quad \tilde{\bar{\nabla}}_{\partial_0}\partial_2 = -\frac{1}{\bar{y}^0}\partial_2 + \frac{\lambda_0}{\bar{y}^0}\partial_3, \\ \tilde{\bar{\nabla}}_{\partial_0}\partial_3 = -\frac{1}{\bar{y}^0}\partial_3 - \frac{\lambda_0}{\bar{y}^0}\partial_2, \quad \tilde{\bar{\nabla}}_{\partial_1}\partial_2 = \frac{\lambda_1}{\bar{y}^0}\partial_3, \quad \tilde{\bar{\nabla}}_{\partial_1}\partial_3 = -\frac{\lambda_1}{\bar{y}^0}\partial_2, \end{array}$$

where ∂_k stands for $\frac{\partial}{\partial\bar{y}^k}$. Let $H \simeq \mathbb{R}$ be the subgroup of $\mathbb{R}H(4)$ given by

$$H = \{(1, \lambda, 0, 0)/\lambda \in \mathbb{R}\}.$$

We take the H-principal bundle

$$\begin{array}{rcl} \mathbb{R}H(4) & \to & \mathbb{R}H(3) \\ (\bar{y}^0, \bar{y}^1, \bar{y}^2, \bar{y}^3) & \mapsto & (\bar{y}^0, \bar{y}^2, \bar{y}^3) \end{array}$$

with mechanical connection form $\omega = d\bar{y}^1$. We have that

$$\tilde{\bar{\nabla}}\omega = \left(\frac{1}{\bar{y}^0} d\bar{y}^0\right) \cdot \omega,$$

where we have identified $\mathfrak{h} \simeq \mathbb{R}$ and $\mathrm{End}(\mathfrak{h}) \simeq \mathbb{R}$. From Theorem 7.2.1, the family of homogeneous structure tensors $\bar{S}^{(\lambda_0,\lambda_1)}$ can be reduced to $\mathbb{R}H(3)$. If (y^0, y^1, y^2) are the standard coordinates of $\mathbb{R}H(3)$, these reduced homogeneous structure tensors form a one-parameter family

$$S^{\lambda_0} = \frac{1}{(y^0)^3}\left(\sum_{k=1}^{2} dy^k \otimes dy^k \wedge dy^0 - \lambda_0 dy^0 \otimes dy^1 \wedge dy^2\right).$$

Note that in the expression of both $\bar{S}^{(\lambda_0,\lambda_1)}$ and S^{λ_0} the first summand is the standard $\mathcal{S}_1$ structure of $\mathbb{R}H(4)$ and $\mathbb{R}H(3)$ respectively. The other summands are of type $\mathcal{S}_2 \oplus \mathcal{S}_3$ since they have null trace, which makes $\bar{S}^{(\lambda_0,\lambda_1)}$ and S^{λ_0} of type $\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3$ in the generic case. In the especial case $\lambda_0 = 0$ we will have a reduction of the generic class $\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3$ to the class $\mathcal{S}_1$.

(b) The Hopf fibration $S^3 \to S^2$

Let $S^3 \subset \mathbb{R}^4 \simeq \mathbb{C}^2$ be the 3-sphere with its standard Riemannian metric with full isometry group $O(4)$. The natural action of $U(2)$ in $\mathbb{C}^2$ defines a transitive and effective action of $U(2)$ on S^3 given by

$$\begin{array}{ccc} U(2) & \hookrightarrow & SO(4) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \begin{pmatrix} \mathrm{Re}(a) & -\mathrm{Im}(a) & \mathrm{Re}(b) & -\mathrm{Im}(b) \\ \mathrm{Im}(a) & \mathrm{Re}(a) & \mathrm{Im}(b) & \mathrm{Re}(b) \\ \mathrm{Re}(c) & -\mathrm{Im}(c) & \mathrm{Re}(d) & -\mathrm{Im}(d) \\ \mathrm{Im}(c) & \mathrm{Re}(c) & \mathrm{Im}(d) & \mathrm{Re}(d) \end{pmatrix} \end{array}.$$

The isotropy group at $\bar{x} = (1,0,0,0) \in S^3$ is

$$\bar{K} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \in U(2)/z \in U(1) \right\}$$

with Lie algebra

$$\bar{\mathfrak{k}} = \mathrm{Span}\left\{ \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \right\}.$$

It is easy to see that the complement

$$\bar{\mathfrak{m}} = \mathrm{Span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\}$$

makes $\mathfrak{u}(2) = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ a reductive decomposition. The rest of complements $\bar{\mathfrak{m}}'$ giving reductive decompositions $\mathfrak{u}(2) = \bar{\mathfrak{m}}' \oplus \bar{\mathfrak{k}}$ are obtained as the graph of $\mathrm{Ad}(\bar{K})$-equivariant maps $\varphi : \bar{\mathfrak{m}} \to \bar{\mathfrak{k}}$. One can check that these decompositions are exhausted by the following one-parameter family of complements

$$\bar{\mathfrak{m}}_\lambda = \mathrm{Span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \right\}, \quad \lambda \in \mathbb{R}.$$

From (7.13), the expression of the homogeneous structure tensor $\bar{S}^\lambda$ associated to each reductive decomposition computed at $T_{\bar{x}}S^3$ is given by

$$(\bar{S}^\lambda)_{\bar{x}} = (\lambda - 1) d\bar{x}^2 \otimes d\bar{x}^3 \wedge d\bar{x}^4 + d\bar{x}^3 \otimes d\bar{x}^2 \wedge d\bar{x}^4 - d\bar{x}^4 \otimes d\bar{x}^2 \wedge d\bar{x}^3, \qquad (7.28)$$

where $(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4)$ is the natural system of coordinates in $\mathbb{R}^4$. Let H be the subgroup of $U(2)$ isomorphic to $U(1)$ given by

$$H = \left\{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} /z \in U(1) \right\}.$$

It is easy to check that H is a normal subgroup of $U(2)$ acting freely on S^3. Reduction by the action of H gives the Hopf fibration $S^3 \to S^2$ with vertical and horizontal subspaces at $\bar{x}$

$$V_{\bar{x}}S^3 = \mathrm{Span}\left\{ \frac{\partial}{\partial \bar{x}_2} \right\}, \quad H_{\bar{x}}S^3 = \mathrm{Span}\left\{ \frac{\partial}{\partial \bar{x}_3}, \frac{\partial}{\partial \bar{x}_4} \right\}.$$

Since all the terms of $\bar{S}^\lambda$ have the vertical factor $d\bar{x}^2$, it is obvious that they all reduce to the structure tensor $S = 0$ on S^2, describing S^2 as a symmetric space. Note that this is what one can expect since S^2 only admits the zero homogeneous structure tensor [60]. For the case $\lambda = 0$ one can compute the transvection algebra associated to $\bar{S}^\lambda$ obtaining the Lie algebra of a Lie group acting transitively by isometries on S^3. As a simple computation shows the holonomy of the connection $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}^0$ is trivial, and one obtains the reductive decomposition $T_eS^3 \oplus \{0\} \simeq \mathfrak{su}(2)$ which describes the action of

$SU(2) \simeq S^3$ on itself. We then have an example of a homogeneous Riemannian structure $\bar{S}^0$ satisfying $\tilde{\tilde{\nabla}}\omega = \alpha \cdot \omega$ as in Theorem 7.2.1 (ω being the mechanical connection form of the Hopf fibration $S^3 \to S^2$), but for which the structure group of the fibration ($H = U(1)$) cannot be seen as a normal subgroup of the group ($\bar{G}' = SU(2)$) obtained from the transvection algebra.

Remark 7.2.6 *There are not more reducible tensors than those described above as the other groups acting transitively on S^3 are $SO(4)$, which has no normal subgroups, and $SU(2) \simeq S^3$. In addition, this procedure can be adapted to Berger 3-spheres, where a family of homogeneous structures is computed in [32]. All reducible structures of this family reduce to $S = 0$ on S^2 as expected.*

(c) The Hopf fibrations $S^7 \to S^4$ and $S^7 \to \mathbb{C}P^3$

The groups acting isometrically and transitively on S^7 (see [53]) are $SO(7)$, $SU(4)$, $Sp(2)Sp(1)$, $U(4)$ and $Sp(2)U(1)$. The first two groups don't have normal subgroups and hence don't fit in the reduction scheme. The group $\bar{G} = Sp(2)Sp(1)$ has the normal subgroup $H = Sp(1) = SU(2)$, which gives the Hopf fibration $S^7 \to S^4$. In this case, a similar computation to the fibration $S^3 \to S^2$ shows that the corresponding homogeneous Riemannian structures in the 7-sphere reduce to the null tensor on S^4, the only homogeneous structure in the four dimensional sphere. We analyze the remaining two groups.

Let Δ^i_j denote the 4×4 complex matrix with 1 in the i-th row and the j-th column and the rest zeros. Let S^7 be the standard 7-sphere as a Riemannian sub-manifold of $\mathbb{C}^4$ with the usual Hermitian inner product. The standard action of the unitary group $U(4)$ on $\mathbb{C}^4$ gives a transitive and effective action by isometries on S^7. The isotropy group $\bar{K}$ at $\bar{x} = (1,0,0,0) \in \mathbb{S}^7$ is isomorphic to $U(3)$ and we can decompose $\mathfrak{u}(4) = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ with

$$\bar{\mathfrak{k}} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{u}(3) \right\}$$

and

$$\bar{\mathfrak{m}} = \mathrm{Span}\{i\Delta^1_1,\, \Delta^1_j - \Delta^j_1, i(\Delta^1_j + \Delta^j_1),\, j = 1,2,3\}.$$

One can check that $\mathfrak{u}(4) = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ is the unique reductive decomposition of $\mathfrak{u}(4)$ with respect to $\bar{\mathfrak{k}}$. From (7.13), identifying $\mathbb{R}^8 \simeq \mathbb{C}^4$ and taking its natural coordinates $(\bar{x}^1, \ldots, \bar{x}^8)$, the homogeneous structure tensor $\bar{S}$ associated to this decomposition at $T_{\bar{x}}S^7$ reads

$$\begin{aligned} \bar{S}_{\bar{x}} &= d\bar{x}^3 \otimes d\bar{x}^2 \wedge d\bar{x}^4 - d\bar{x}^4 \otimes d\bar{x}^2 \wedge d\bar{x}^3 + d\bar{x}^5 \otimes d\bar{x}^2 \wedge d\bar{x}^6 \\ &\quad - d\bar{x}^6 \otimes d\bar{x}^2 \wedge d\bar{x}^5 + d\bar{x}^7 \otimes d\bar{x}^2 \wedge d\bar{x}^8 - d\bar{x}^8 \otimes d\bar{x}^2 \wedge d\bar{x}^7. \end{aligned} \tag{7.29}$$

As a simple computation shows, this tensor belongs to the class $\mathcal{S}_2 \oplus \mathcal{S}_3$.

Let H be the subgroup of $U(4)$ isomorphic to $U(1)$ given by

$$H = \{z \cdot Id / z \in U(1)\}$$

where Id is the 4×4 identity matrix. It is obvious that H is a normal subgroup of $U(4)$ the action of which on S^7 is free. The reduction of S^7 by the action of H gives the Hopf fibration $S^7 \to \mathbb{C}P^3$ from which the complex projective space inherits the Fubiny-Study metric. The vertical and horizontal subspaces at $\bar{x}$ are

$$V_{\bar{x}}S^7 = \mathrm{Span}\left\{\frac{\partial}{\partial \bar{x}_2}\right\}, \quad H_{\bar{x}}S^7 = \mathrm{Span}\left\{\frac{\partial}{\partial \bar{x}_3}, \ldots, \frac{\partial}{\partial \bar{x}_8}\right\}.$$

As in the Hopf fibration $S^3 \to S^2$, the homogeneous structure tensor $\bar{S}$ reduces to $S = 0$, describing

$$\mathbb{C}P^3 = \frac{U(4)}{U(3) \times U(1)}$$

as a symmetric space.

Let $\mathbb{H}$ denote the quaternion algebra, we now see the 7-sphere

$$S^7 = \left\{ \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathbb{H}^2 \,\middle|\, |q_1|^2 + |q_2|^2 = 1 \right\}$$

as a Riemannian submanifold of $\mathbb{H}^2$ with the standard quaternion inner product. The group $Sp(2)U(1)$ acts on $\mathbb{H}^2$ by

$$(A, z) \cdot \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = A \begin{pmatrix} q_1 \overline{z} \\ q_2 \overline{z} \end{pmatrix}, \qquad \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathbb{H}^2, A \in Sp(2), z \in U(1),$$

where $\overline{z}$ stands for complex conjugation. This action restricts to a transitive and effective action by isometries on S^7. The isotropy group at $\bar{x} = (1, 0) \in S^7$ is

$$\bar{K} = \left\{ \left(\begin{pmatrix} z & 0 \\ 0 & q \end{pmatrix}, z \right) \,\middle|\, q \in Sp(1), z \in U(1) \right\} / \mathbb{Z}_2,$$

which is isomorphic to $Sp(1)U(1)$. The Lie algebra of $Sp(2)U(1)$ is $\mathfrak{sp}(2) \oplus \mathfrak{u}(1)$ where

$$\mathfrak{sp}(2) = \mathrm{Span} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} \right\}$$

and $\mathfrak{u}(1) = \mathrm{Span}\{i\}$. The isotropy algebra is thus

$$\bar{\mathfrak{k}} = \mathrm{Span} \left\{ \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} + i, \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} \right\}.$$

Taking

$$\bar{\mathfrak{m}} = \mathrm{Span} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \right\}$$

we have that $\mathfrak{sp}(2) \oplus \mathfrak{u}(1) = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ is a reductive decomposition. All other reductive decompositions associated to $\mathfrak{sp}(2) \oplus \mathfrak{u}(1)$ and $\bar{\mathfrak{k}}$ are given by a one-parameter family of complements $\bar{\mathfrak{m}}_\lambda$, $\lambda \in \mathbb{R}$, which are the graph of the $\mathrm{Ad}(\bar{K})$-equivariant maps $\varphi_\lambda : \bar{\mathfrak{m}} \to \bar{\mathfrak{k}}$, where φ_λ maps $\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$ to $\lambda \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} + \lambda i$ and the rest of elements of the basis to zero. Identifying $\mathbb{H}^2 \simeq \mathbb{R}^8$, the homogeneous structure tensor $\bar{S}^\lambda$ associated to each reductive decomposition $\mathfrak{sp}(2) \oplus \mathfrak{u}(1) = \bar{\mathfrak{m}}_\lambda \oplus \bar{\mathfrak{k}}$ is computed at $T_{\bar{x}} S^7$ as

$$\begin{aligned}
(\bar{S}^\lambda)_{\bar{x}} \;=\;& d\bar{x}^5 \otimes d\bar{x}^2 \wedge d\bar{x}^6 + d\bar{x}^5 \otimes d\bar{x}^3 \wedge d\bar{x}^7 + d\bar{x}^5 \otimes d\bar{x}^4 \wedge d\bar{x}^8 \\
& -\lambda d\bar{x}^2 \otimes d\bar{x}^5 \wedge d\bar{x}^6 + (1 + 2\lambda) d\bar{x}^2 \otimes d\bar{x}^3 \wedge d\bar{x}^4 + \lambda d\bar{x}^2 \otimes d\bar{x}^7 \wedge d\bar{x}^8 \\
& +d\bar{x}^6 \otimes d\bar{x}^5 \wedge d\bar{x}^2 + d\bar{x}^6 \otimes d\bar{x}^3 \wedge d\bar{x}^8 - d\bar{x}^6 \otimes d\bar{x}^4 \wedge d\bar{x}^7 \\
& +d\bar{x}^3 \otimes d\bar{x}^2 \wedge d\bar{x}^4 + d\bar{x}^4 \otimes d\bar{x}^2 \wedge d\bar{x}^3 \\
& -d\bar{x}^7 \otimes d\bar{x}^3 \wedge d\bar{x}^5 - d\bar{x}^7 \otimes d\bar{x}^2 \wedge d\bar{x}^8 + d\bar{x}^7 \otimes d\bar{x}^4 \wedge d\bar{x}^6 \\
& -d\bar{x}^8 \otimes d\bar{x}^4 \wedge d\bar{x}^5 + d\bar{x}^8 \otimes d\bar{x}^2 \wedge d\bar{x}^7 - d\bar{x}^8 \otimes d\bar{x}^3 \wedge d\bar{x}^6.
\end{aligned}$$

Let $H = \{(Id, w) / \, w \in U(1)\} \subset Sp(2)U(1)$, where Id is the identity of $Sp(2)$, it is easy to see that H is a normal subgroup of $Sp(2)U(1)$ isomorphic to $U(1)$. Reduction by the action of H gives again the Hopf fibration $\pi : S^7 \to \mathbb{C}P^3$ with $\pi(\bar{x}) = [1 : 0 : 0 : 0] \in \mathbb{C}P^3$. The vertical and horizontal subspaces of π at $\bar{x}$ are

$$V_{\bar{x}} S^7 = \mathrm{Span} \left\{ \frac{\partial}{\partial \bar{x}_2} \right\}, \quad H_{\bar{x}} S^7 = \mathrm{Span} \left\{ \frac{\partial}{\partial \bar{x}_3}, \ldots, \frac{\partial}{\partial \bar{x}_8} \right\}$$

Let $(t^1,\dots,t^6):\mathbb{C}P^3-\{z_0=0\}\to\mathbb{R}^6$ be the coordinate system around $x=[1:0:0:0]$ given by

$$[z_0\colon z_1\colon z_2\colon z_3]\mapsto\left(\mathrm{Re}\left(\tfrac{z_1}{z_0}\right),\mathrm{Im}\left(\tfrac{z_1}{z_0}\right),\mathrm{Re}\left(\tfrac{z_2}{z_0}\right),\mathrm{Im}\left(\tfrac{z_2}{z_0}\right),\mathrm{Re}\left(\tfrac{z_3}{z_0}\right),\mathrm{Im}\left(\tfrac{z_3}{z_0}\right)\right).$$

The reduced homogeneous structure tensor S at $T_x\mathbb{C}P^3$ is

$$\begin{aligned}S_x &= dt^3\otimes dt^1\wedge dt^5+dt^3\otimes dt^2\wedge dt^6+dt^4\otimes dt^1\wedge dt^6-dt^4\otimes dt^2\wedge dt^5\\ &\quad+dt^5\otimes dt^2\wedge dt^4-dt^5\otimes dt^1\wedge dt^3-dt^6\otimes dt^2\wedge dt^3-dt^6\otimes dt^1\wedge dt^4.\end{aligned}$$

It is easy to check that $\bar{S}^\lambda$ is of type $\mathcal{S}_2\oplus\mathcal{S}_3$ for all $\lambda\in\mathbb{R}$ which is not $\mathcal{S}_2$ nor $\mathcal{S}_3$ for any λ, and S is also a strict $\mathcal{S}_2\oplus\mathcal{S}_3$ structure. Note that in the latter and the former example the class $\mathcal{S}_2\oplus\mathcal{S}_3$ is preserved by the reduction procedure. This fact is expected from Proposition 7.2.4, since the fibers of the Hopf fibration are totally geodesic and in particular minimal Riemannian submanifolds of S^7.

7.3 Application to cosymplectic and Sasakian homogeneous structures of linear type

We recall that a vector field $\bar{X}$ on $\bar{M}$ is said to be *regular* if it is nowhere vanishing. In that case every point $\bar{x}\in\bar{M}$ has a neighborhood U with coordinates $(x^1,\dots,x^{2n+1})$ whose intersection with any integral curve of $\bar{X}$ is given by $x^1=const.,\dots,x^{2n+1}=const.$ In addition $\bar{X}$ is called *strictly regular* if all the integral curves are homeomorphic.

Definition 7.3.1 *An almost contact structure (ϕ,ξ,η) is*

1. *(strictly) regular if the vector field ξ is (strictly) regular, and*
2. *invariant if ϕ and η are invariant by the 1-parameter group generated by ξ.*

Remark 7.3.2 *It is easy to prove that in the cosymplectic and Sasakian cases the invariance property is automatically satisfied since $\mathcal{L}_\xi\phi=0=\mathcal{L}_\xi\eta$.*

Let (ϕ,ξ,η) be a regular almost contact structure on a manifold $\bar{M}$. We consider M the orbit space defined by ξ, which is a differentiable manifold, and the natural projection $\pi:\bar{M}\to M$. The following results were obtained in [51].

Theorem 7.3.3 *Let (ϕ,ξ,η) be a strictly regular invariant almost contact structure on $\bar{M}$, and let H be the 1-parameter group generated by ξ. Then*

1. *$\pi:\bar{M}\to M$ is a principal H-bundle over M, and*
2. *η is a connection form on $\pi:\bar{M}\to M$.*

Hereafter we will only consider strictly regular invariant almost contact structures.

Theorem 7.3.4 *The $(1,1)$-tensor field defined by*

$$J_xX=\pi_*\left(\phi X^H\right),\qquad X\in T_xM,$$

where X^H denotes the horizontal lift of X with respect to the connection η, is an almost complex structure on M. Moreover, $[\phi,\phi]=0$ if and only if J is a complex structure and η is a flat connection. On the other hand, $[\phi,\phi]+2\eta\otimes\xi=0$ if and only if J is a complex structure and $\Theta(JX,JY)=\Theta(X,Y)$, where $\pi^*\Theta=d\eta$.

We now consider an almost contact metric structure $(\phi, \xi, \eta, \bar{g})$ on $\bar{M}$. Note that since $\eta = \varepsilon\xi^\flat$, the connection η coincides with the mechanical connection on $\pi : \bar{M} \to M$ with respect to $\bar{g}$.

Theorem 7.3.5 *[51] Let $(\phi, \xi, \eta, \bar{g})$ be an almost contact metric structure on $\bar{M}$, and let g and J be the reduced metric and almost complex structure on M. If $(\phi, \xi, \eta, \bar{g})$ is cosymplectic or Sasakian then (M, g, J) is pseudo-Kähler.*

Let $\bar{S}$ be a homogeneous cosymplectic or Sasakian structure invariant by the 1-parameter group generated by ξ. Let $\bar{\widetilde{\nabla}}$ be the linear connection associated to $\bar{S}$, since η defines the mechanical connection on π and $\bar{\widetilde{\nabla}}\eta = 0$, we are in the situation of Theorem 7.2.1 so that $S_X Y = \pi_*(\bar{S}_{X^H} Y^H)$ is a homogenous pseudo-Riemannian structure on M.

Proposition 7.3.6 *If $\bar{S}$ is a homogeneous cosymplectic or Sasakian structure on $\bar{M}$, then S is a homogeneous pseudo-Kähler structure on M.*

Proof. Let $\widetilde{\nabla} = \nabla - S$, where ∇ is the Levi-Civita connection of g. Then $\widetilde{\nabla}_X Y = \pi_*(\bar{\widetilde{\nabla}}_{X^H} Y^H)$. Since $\eta(\phi(\bar{X})) = 0$ we have that $\phi(\bar{X})$ is horizontal for all $\bar{X} \in \mathfrak{X}(\bar{M})$. We thus have

$$\begin{aligned}
\left(\widetilde{\nabla}_X J\right) Y &= \widetilde{\nabla}_X(JY) - J\left(\widetilde{\nabla}_X Y\right) \\
&= \pi_*\left(\bar{\widetilde{\nabla}}_{X^H}(JY)^H\right) - \pi_*\left(\phi\left(\bar{\widetilde{\nabla}}_{X^H} Y^H\right)\right) \\
&= \pi_*\left(\bar{\widetilde{\nabla}}_{X^H}\left(\phi Y^H\right) - \phi\left(\bar{\widetilde{\nabla}}_{X^H} Y^H\right)\right) \\
&= \pi_*\left(\left(\bar{\widetilde{\nabla}}_{X^H}\phi\right) Y^H\right) \\
&= 0
\end{aligned}$$

for every $X, Y \in \mathfrak{X}(M)$, whence $\widetilde{\nabla} J = 0$. ∎

Proposition 7.3.7 *If $\bar{S}$ is of linear type then S is of linear type. Moreover, if $\bar{S}$ is non-degenerate (resp. degenerate) then S is non-degenerate (resp. degenerate).*

Proof. Let $\bar{\chi}$ and $\bar{\zeta}$ be the basic vector fields determining $\bar{S}$. Since $\bar{S}$ is invariant by the one parameter group generated by ξ, it is easy to prove that so are $\bar{\chi}$ and $\bar{\zeta}$. In particular there are vector fields χ and ζ on M such that $\pi_*(\bar{\chi}) = \chi$ and $\pi_*(\bar{\zeta}) = \zeta$. A simple inspection shows that $S_X Y = \pi_*(\bar{S}_{X^H} Y^H)$ takes the form (4.2) for the vector fields χ and ζ, so that S is of linear type. It is obvious that $\bar{g}(\bar{\chi}, \bar{\chi}) = g(\chi, \chi)$. ∎

In view of the previous Propositions, we can study invariant homogeneous cosymplectic and Sasakian structures of linear type via the reduction procedure and making use of the results on homogeneous pseudo-Kähler structures of linear type obtained in Chapter 5.

Proposition 7.3.8 *Let $(\bar{M}, \bar{g}, \phi, \xi, \eta)$ be a cosymplectic manifold of dimension $2n + 5$ admitting an invariant homogeneous cosymplectic structure of linear type $\bar{S}$.*

1. *If $\bar{S}$ is non-degenerate, then $\bar{\zeta} = 0$ and $(\bar{M}, \bar{g}, \phi, \xi, \eta)$ is of constant ϕ-sectional curvature $k = -4g(\bar{\chi}, \bar{\chi})$.*
2. *If $\bar{S}$ is degenerate, then $\bar{\zeta} = \lambda\bar{\chi}$ with $\lambda \in \{0, 1/2\}$, and $(\bar{M}, \bar{g}, \phi, \xi, \eta)$ is locally isometric to $\mathbb{C}^{n+2} \times \mathbb{R}$ with metric $\bar{g} = g + \varepsilon dt^2$ and its standard cosymplectic structure, where g is the metric on $\mathbb{C}^{n+2}$ given by* (5.25) *for $\epsilon = -1$.*

Proof.

1. Since the reduced homogeneous structure S is a non-degenerate homogeneous pseudo-Kähler structure of linear type, we have by Proposition 5.1.2 that (M,g,J) has constant holomorphic sectional curvature $c=-4g(\pi_*\chi,\pi_*\chi)$. Let $\bar{X}$ be a vector in the contact distribution, by O'Neill's formulas for pseudo-Riemannian submersions
$$\bar{K}\left(\bar{X},\phi\bar{X}\right)=K\left(\pi_*\bar{X},\pi_*\phi\bar{X}\right)+\frac{3}{4}\left([\bar{X},\phi\bar{X}]^V\right)^2,$$
where $\bar{K}$ and K denote sectional curvatures on $\bar{M}$ and M respectively, and $\bar{X}$ is unitary. As $(\bar{M},\bar{g},\phi,\xi,\eta)$ is cosymplectic, the curvature of the connection η vanishes so that $[\bar{X},\phi\bar{X}]^V=0$. We thus conclude that
$$\begin{aligned}\bar{K}\left(\bar{X},\phi\bar{X}\right)&=K\left(\pi_*\bar{X},\pi_*\phi\bar{X}\right)=K\left(\pi_*\bar{X},J\pi_*\bar{X}\right)\\&=-4g(\pi_*\bar{\chi},\pi_*\bar{\chi})=-4\bar{g}(\bar{\chi},\bar{\chi}).\end{aligned}$$
Note that $\bar{g}(\bar{\chi},\bar{\chi})$ is constant since $\tilde{\bar{\nabla}}\bar{g}=0$, and $\tilde{\bar{\nabla}}\bar{S}=0$ implies $\tilde{\bar{\nabla}}\bar{\chi}=0$, where $\tilde{\bar{\nabla}}$ is the ASK-connection associated to $\bar{S}$.

2. Since S is a degenerate homogeneous pseudo-Kähler structure of linear type determined by the vector fields $\pi_*\bar{\chi}$ and $\pi_*\bar{\zeta}$, we have by Corollary 5.2.3 that $\pi_*\bar{\zeta}=\lambda\pi_*\bar{\chi}$ with $\lambda\in\{0,1/2\}$. This implies $\bar{\zeta}=\lambda\bar{\chi}$, as $\bar{\zeta}$ and $\bar{\chi}$ are basic by definition.

   Let now H be the one parameter group generated by ξ. Since $(\bar{M},\bar{g},\phi,\xi,\eta)$ is cosymplectic, the curvature of the connection η on $\pi:\bar{M}\to M$ vanishes. This implies that for every $\bar{p}\in\bar{M}$ we can choose a trivialization $\Psi:\pi^{-1}(U)\to U\times H$ with $\Psi(\bar{p})=(\pi(\bar{p}),e)$ such that $\Psi_*(H_{\bar{q}}\bar{M})=T_{\pi(\bar{p})}U+\{0\}$ for all $\bar{q}\in\pi^{-1}(U)$, where e is the neutral element of H and $H_{\bar{q}}\bar{M}$ is the horizontal subspace at $\bar{q}$ with respect to η, which coincides with the contact distribution $\mathrm{Span}\{\xi\}^\perp$. Taking a coordinate $t\in(-\delta,\delta)$ on a neighborhood of H around e, we obtain a diffeomorphism $\Psi:V\to U\times(-\delta,\delta)$, where V is a certain neighborhood around $\bar{p}$. By construction we have that $(\Psi^{-1})^*(\bar{g})=g_U+\varepsilon dt^2$, where g_U is the reduced metric g on M restricted to U. Reducing U if necessary we have that g_U is holomorphically isometric to an open set W of $\mathbb{C}^{n+2}$ with metric (5.25) for $\epsilon=-1$. Finally, it is easy to see that the cosymplectic structure on $\bar{M}$ transforms by Ψ into the standard cosymplectic structure of $\mathbb{C}^{n+2}\times\mathbb{R}$ restricted to the open set $W\times(-\delta,\delta)$.

■

Corollary 7.3.9 *Let $(\bar{M},\bar{g},\phi,\xi,\eta)$ be a cosymplectic manifold of dimension $2n+5$ admitting an invariant degenerate homogeneous cosymplectic structure of linear type. Then $\bar{g}$ is Ricci-flat and the holonomy algebra is given by the one dimensional Lie algebra*
$$\mathfrak{hol}(\bar{g})\cong\mathbb{R}\begin{pmatrix} i & i & 0\\ -i & -i & 0\\ 0 & 0 & 0_n\end{pmatrix}+\{0\}\subset\mathfrak{su}(1,1)+\{0\}\subset\mathrm{Lie}(U(p,q)\times 1),$$
with $p+q=n+2$.

Proof. This result follows immediately from the fact that g is Ricci-flat and has holonomy algebra
$$\mathfrak{hol}(g)\cong\mathbb{R}\begin{pmatrix} i & i & 0\\ -i & -i & 0\\ 0 & 0 & 0_n\end{pmatrix}.$$

■

Proposition 7.3.10 *Let $(\bar{M}, \bar{g}, \phi, \xi, \eta)$ be a Sasakian manifold admitting an invariant homogeneous Sasakian structure of linear type $\bar{S}$.*

1. *If $\bar{S}$ is non-degenerate, then $\bar{\zeta} = 0$ and $(\bar{M}, \bar{g}, \phi, \xi, \eta)$ is of constant ϕ-sectional curvature $k = -4g(\bar{\chi}, \bar{\chi}) + 3$.*

2. *If $\bar{S}$ is degenerate, then $\bar{\zeta} = \lambda\bar{\chi}$ with $\lambda \in \{0, 1/2\}$, and there is a set of coordinates $\{z^1, z^2, w^1, w^2, x^a, y^a, t\}$ such that the metric takes the form*

$$\bar{g} = g + \varepsilon\eta dt,$$

where g is given by (5.25) for $\epsilon = -1$.

Proof.

1. Since the reduced homogeneous structure S is a non-degenerate homogeneous pseudo-Kähler structure of linear type, we have by Proposition 5.1.2 that (M, g, J) has constant holomorphic sectional curvature $c = -4g(\pi_*\bar{\chi}, \pi_*\bar{\chi})$. Let $\bar{X}$ be a vector in the contact distribution, by O'Neill's formulas for pseudo-Riemannian submersions

$$\bar{K}\left(\bar{X}, \phi\bar{X}\right) = K\left(\pi_*\bar{X}, \pi_*\phi\bar{X}\right) + \frac{3}{4}\left([\bar{X}, \phi\bar{X}]^V\right)^2,$$

where $\bar{K}$ and K denote sectional curvatures on $\bar{M}$ and M respectively, and $\bar{X}$ is unitary. As $(\bar{M}, \bar{g}, \phi, \xi, \eta)$ is Sasakian, the curvature form of the connection η is given by $d\eta = \Phi$, so that

$$[\bar{X}, \phi\bar{X}]^V = 2d\eta(\bar{X}, \phi\bar{X}) = 2\Phi(\bar{X}, \phi\bar{X}).$$

We thus conclude that

$$\begin{aligned} \bar{K}\left(\bar{X}, \phi\bar{X}\right) &= K\left(\pi_*\bar{X}, J\pi_*\bar{X}\right) + 3\Phi(\bar{X}, \phi\bar{X})^2 \\ &= -4g(\pi_*\bar{\chi}, \pi_*\bar{\chi}) + 3 = -4\bar{g}(\bar{\chi}, \bar{\chi}) + 3. \end{aligned}$$

Note that by the same reason as before $\bar{g}(\bar{\chi}, \bar{\chi})$ is constant.

2. Since S is a degenerate homogeneous pseudo-Kähler structure of linear type determined by the vector fields $\pi_*\bar{\chi}$ and $\pi_*\bar{\zeta}$, we have by Corollary 5.2.3 that $\pi_*\bar{\zeta} = \lambda\pi_*\bar{\chi}$ with $\lambda \in \{0, 1/2\}$, so that $\bar{\zeta} = \lambda\bar{\chi}$. Let now H be the 1-parameter group generated by ξ. Let $\bar{p} \in \bar{M}$ we consider a trivialization of the principal bundle $\pi : \bar{M} \to M$, i.e., we consider a diffeomorphism $\Psi : \pi^{-1}(U) \to U \times H$, where $\Psi(\bar{p}) = (\pi(\bar{p}), e)$ and e is the neutral element of H. Let $\widetilde{\xi} = \Psi_{*,\bar{p}}(\xi_{\bar{p}}) \in \mathfrak{h}$, there is an open interval $(-\delta, \delta)$ such that $f : (-\delta, \delta) \to H$ given by $f(t) = \exp(t\widetilde{\xi})$ is a diffeomorphism onto its image. We thus consider the map

$$F : U \times (-\delta, \delta) \overset{\mathrm{id}\times f}{\to} U \times H \overset{\Psi^{-1}}{\to} \pi^{-1}(U),$$

which is a diffeomorphism onto a certain neighborhood of $\bar{p}$. Then, the pullback of the metric $\bar{g}$ by F is $F^*\bar{g} = g_U + \varepsilon F^*\eta dt$, where g_U is the reduced metric g restricted to U. Reducing U if necessary we can take coordinates $\{z^1, z^2, w^1, w^2, x^a, y^a\}$ on U such that g_U is expressed as (5.25) for $\epsilon = -1$. We have thus constructed coordinates $\{z^1, z^2, w^1, w^2, x^a, y^a, t\}$ around $\bar{p}$ with respect to which $\bar{g} = g + \varepsilon\eta dt$ with g given by (5.25) for $\epsilon = -1$.

■

Note that due to the non-integrability of the contact distribution, in the previous proof $\Psi_*(H_{\bar{q}}\bar{M}) \neq T_{\pi(\bar{p})}U + \{0\}$ for general $\bar{q} \in \pi^{-1}(U)$. This means that $F^*\eta \neq dt$, so that the components of η with respect to $\{z^1, z^2, w^1, w^2, x^a, y^a, t\}$ don't identically vanish. Actually, if we write

$$\eta = \eta_{z^1} dz^1 + \eta_{z^2} dz^2 + \eta_{w^1} dw^1 + \eta_{w^2} dw^2 + \eta_{x^a} dx^a + \eta_{y^a} dy^a + dt,$$

we have to impose that $d\eta = \Phi$. Since $\pi^* \omega = \Phi$, where ω is the symplectic form associated to g and J we have that

$$\partial_\beta \eta_\alpha - \partial_\alpha \eta_\alpha = \omega_{\alpha\beta}, \qquad \alpha, \beta = z^1, z^2, w^1, w^2, x^a, y^a,$$

with

$$\omega = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & -b & 0 & \dots & 0 \\ 1 & 0 & b & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & & \Sigma & \\ 0 & 0 & 0 & 0 & & & \end{pmatrix},$$

and

$$\Sigma = \operatorname{diag}\left(\begin{pmatrix} 0 & -\varepsilon^a \\ \varepsilon^a & 0 \end{pmatrix}, a = 1, \dots, n \right).$$

Remark 7.3.11 *The previous results only apply to invariant homogeneous structures, and therefore they do not fully characterize homogeneous cosymplectic and Sasakian structures of linear type. Nevertheless, the reduction procedure has proved to be a powerful tool in the study of these kind of structures, giving a head start for the study of this topic.*

Chapter 8

Appendix: Computations concerning formula (5.17)

We start with formula

$$R_{\xi U} = -2a\epsilon f\theta(JU)\theta \wedge (\theta \circ J), \tag{8.1}$$

and apply

$$(\nabla_X R)_{YZWU} = -R_{S_X YZWU} - R_{YS_X ZWU} - R_{YZS_X WU} - R_{YZWS_X U},$$

that is,

$$\begin{aligned}(\nabla_X R)_{YZWU} &= -g(X,Y)R_{\xi ZWU} + g(\xi,Y)R_{XZWU} - \epsilon g(X,JY)R_{J\xi ZWU}\\ &+ \epsilon g(\xi,JY)R_{JXZWU} - g(X,Z)R_{Y\xi WU} + g(\xi,Z)R_{YXWU}\\ &- \epsilon g(X,JZ)R_{YJ\xi WU} + \epsilon g(\xi,JZ)R_{YJXWU} - g(X,W)R_{YZ\xi U}\\ &+ g(\xi,W)R_{YZXU} - \epsilon g(X,JW)R_{YZJ\xi U} + \epsilon g(\xi,JW)R_{YZJXU}\\ &- g(X,U)R_{YZW\xi} + g(\xi,U)R_{YZWX} - \epsilon g(X,JU)R_{YZWJ\xi}\\ &+ \epsilon g(\xi,JU)R_{YZWJX}.\end{aligned}$$

This gives

$$\begin{aligned}\underset{XYZ}{\mathfrak{S}} (\nabla_X R)_{YZWU} &= -g(X,Y)R_{\xi ZWU} + g(\xi,Y)R_{XZWU} - \epsilon g(X,JY)R_{J\xi ZWU}\\ &+ \epsilon g(\xi,JY)R_{JXZWU} - g(X,Z)R_{Y\xi WU} + g(\xi,Z)R_{YXWU}\\ &- \epsilon g(X,JZ)R_{YJ\xi WU} + \epsilon g(\xi,JZ)R_{YJXWU} - g(X,W)R_{YZ\xi U}\\ &+ g(\xi,W)R_{YZXU} - \epsilon g(X,JW)R_{YZJ\xi U} + \epsilon g(\xi,JW)R_{YZJXU}\\ &- g(X,U)R_{YZW\xi} + g(\xi,U)R_{YZWX} - \epsilon g(X,JU)R_{YZWJ\xi}\\ &+ \epsilon g(\xi,JU)R_{YZWJX} - g(Y,Z)R_{\xi XWU} + g(\xi,Z)R_{YXWU}\\ &- \epsilon g(Y,JZ)R_{J\xi XWU} + \epsilon g(\xi,JZ)R_{JYXWU} - g(Y,X)R_{Z\xi WU}\\ &+ g(\xi,X)R_{ZYWU} - \epsilon g(Y,JX)R_{ZJ\xi WU} + \epsilon g(\xi,JX)R_{ZJYWU}\\ &- g(Y,W)R_{ZX\xi U} + g(\xi,W)R_{ZXYU} - \epsilon g(Y,JW)R_{ZXJ\xi U}\\ &+ \epsilon g(\xi,JW)R_{ZXJYU} - g(Y,U)R_{ZXW\xi} + g(\xi,U)R_{ZXWY}\\ &- \epsilon g(Y,JU)R_{ZXWJ\xi} + \epsilon g(\xi,JU)R_{ZXWJY} - g(Z,X)R_{\xi YWU}\\ &+ g(\xi,X)R_{ZYWU} - \epsilon g(Z,JX)R_{J\xi YWU} + \epsilon g(\xi,JX)R_{JZYWU}\\ &- g(Z,Y)R_{X\xi WU} + g(\xi,Y)R_{XZWU} - \epsilon g(Z,JY)R_{XJ\xi WU}\\ &+ \epsilon g(\xi,JY)R_{XJZWU} - g(Z,W)R_{XY\xi U} + g(\xi,W)R_{XYZU}\\ &- \epsilon g(Z,JW)R_{XYJ\xi U} + \epsilon g(\xi,JW)R_{XYJZU} - g(Z,U)R_{XYW\xi}\\ &+ g(\xi,U)R_{XYWZ} - \epsilon g(Z,JU)R_{XYWJ\xi} + \epsilon g(\xi,JU)R_{XYWJZ}.\end{aligned}$$

Making use of first Bianchi's identity, this expression simplifies to

$$\begin{aligned}
\mathfrak{S}_{XYZ} (\nabla_X R)_{YZWU} &= g(\xi, Y) R_{XZWU} - \epsilon g(X, JY) R_{J\xi ZWU} + g(\xi, Z) R_{YXWU} \\
&- \epsilon g(X, JZ) R_{YJ\xi WU} - g(X, W) R_{YZ\xi U} - \epsilon g(X, JW) R_{YZJ\xi U} \\
&- g(X, U) R_{YZW\xi} - \epsilon g(X, JU) R_{YZWJ\xi} + g(\xi, Z) R_{YXWU} \\
&- \epsilon g(Y, JZ) R_{J\xi XWU} + g(\xi, X) R_{ZYWU} - \epsilon g(Y, JX) R_{ZJ\xi WU} \\
&- g(Y, W) R_{ZX\xi U} - \epsilon g(Y, JW) R_{ZXJ\xi U} - g(Y, U) R_{ZXW\xi} \\
&- \epsilon g(Y, JU) R_{ZXWJ\xi} + g(\xi, X) R_{ZYWU} - \epsilon g(Z, JX) R_{J\xi YWU} \\
&+ g(\xi, Y) R_{XZWU} - \epsilon g(Z, JY) R_{XJ\xi WU} - g(Z, W) R_{XY\xi U} \\
&- \epsilon g(Z, JW) R_{XYJ\xi U} - g(Z, U) R_{XYW\xi} - \epsilon g(Z, JU) R_{XYWJ\xi}.
\end{aligned}$$

As $\mathfrak{S}_{XYZ}(\nabla_X R)_{YZWU} = 0$ we have

$$\begin{aligned}
2 \mathfrak{S}_{XYZ} g(\xi, X) R_{YZWU} &= -2\epsilon g(X, JY) R_{J\xi ZWU} - 2\epsilon g(X, JZ) R_{YJ\xi WU} \\
&- 2\epsilon g(Y, JZ) R_{J\xi XWU} - g(X, W) R_{YZ\xi U} \\
&- \epsilon g(X, JW) R_{YZJ\xi U} - g(Y, W) R_{ZX\xi U} \\
&- \epsilon g(Y, JW) R_{ZXJ\xi U} - g(Z, W) R_{XY\xi U} \\
&- \epsilon g(Z, JW) R_{XYJ\xi U} - g(X, U) R_{YZW\xi} \\
&- \epsilon g(X, JU) R_{YZWJ\xi} - g(Y, U) R_{ZXW\xi} \\
&- \epsilon g(Y, JU) R_{ZXWJ\xi} - g(Z, U) R_{XYW\xi} \\
&- \epsilon g(Z, JU) R_{XYWJ\xi}.
\end{aligned}$$

Substituting (8.1) we obtain

$$\begin{aligned}
\mathfrak{S}_{XYZ} g(\xi, X) R_{YZWU} &= -\epsilon a f \{2g(X, JY)\theta(Z) [\theta(W)\theta(JU) - \theta(U)\theta(JW)] \\
&+ 2g(Y, JZ)\theta(X) [\theta(W)\theta(JU) - \theta(U)\theta(JW)] \\
&+ 2g(Z, JX)\theta(Y) [\theta(W)\theta(JU) - \theta(U)\theta(JW)] \\
&- g(X, W)\theta(JU) [\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)] \\
&- g(Y, W)\theta(JU) [\theta(Z)\theta(JX) - \theta(X)\theta(JZ)] \\
&- g(Z, W)\theta(JU) [\theta(X)\theta(JY) - \theta(Y)\theta(JX)] \\
&+ g(X, JW)\theta(U) [\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)] \\
&+ g(Y, JW)\theta(U) [\theta(Z)\theta(JX) - \theta(X)\theta(JZ)] \\
&+ g(Z, JW)\theta(U) [\theta(X)\theta(JY) - \theta(Y)\theta(JX)] \\
&+ g(X, U)\theta(JW) [\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)] \\
&+ g(Y, U)\theta(JW) [\theta(Z)\theta(JX) - \theta(X)\theta(JZ)] \\
&+ g(Z, U)\theta(JW) [\theta(X)\theta(JY) - \theta(Y)\theta(JX)] \\
&- g(X, JU)\theta(W) [\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)] \\
&- g(Y, JU)\theta(W) [\theta(Z)\theta(JX) - \theta(X)\theta(JZ)] \\
&- g(Z, JU)\theta(W) [\theta(X)\theta(JY) - \theta(Y)\theta(JX)]\} . \qquad (8.2)
\end{aligned}$$

Switching Y and W, and Z and U, the previous formula becomes

$$\begin{aligned}
\underset{XYZ}{\mathfrak{S}}\, g(\xi,X)R_{WUYZ} = -\epsilon af\,\{&2g(X,JW)\theta(U)\,[\theta(Y)\theta(JZ)-\theta(Z)\theta(JY)]\\
&+2g(W,JU)\theta(X)\,[\theta(Y)\theta(JZ)-\theta(Z)\theta(JY)]\\
&+2g(U,JX)\theta(W)\,[\theta(Y)\theta(JZ)-\theta(Z)\theta(JY)]\\
&-g(X,Y)\theta(JZ)\,[\theta(W)\theta(JU)-\theta(U)\theta(JW)]\\
&-g(Y,W)\theta(JZ)\,[\theta(U)\theta(JX)-\theta(X)\theta(JU)]\\
&-g(U,Y)\theta(JZ)\,[\theta(X)\theta(JW)-\theta(W)\theta(JX)]\\
&+g(X,JY)\theta(Z)\,[\theta(W)\theta(JU)-\theta(U)\theta(JW)]\\
&+g(W,JY)\theta(Z)\,[\theta(U)\theta(JX)-\theta(X)\theta(JU)]\\
&+g(U,JY)\theta(Z)\,[\theta(X)\theta(JW)-\theta(W)\theta(JX)]\\
&+g(X,Z)\theta(JY)\,[\theta(W)\theta(JU)-\theta(U)\theta(JW)]\\
&+g(W,Z)\theta(JY)\,[\theta(U)\theta(JX)-\theta(X)\theta(JU)]\\
&+g(Z,U)\theta(JY)\,[\theta(X)\theta(JW)-\theta(W)\theta(JX)]\\
&-g(X,JZ)\theta(Y)\,[\theta(W)\theta(JU)-\theta(U)\theta(JW)]\\
&-g(W,JZ)\theta(Y)\,[\theta(U)\theta(JX)-\theta(X)\theta(JU)]\\
&-g(U,JZ)\theta(Y)\,[\theta(X)\theta(JW-\theta(W)\theta(JX)]\}\,. \qquad (8.3)
\end{aligned}$$

We thus have

$$\begin{aligned}
(\nabla_X R)_{YZWU} = &\,2a\epsilon fg(X,Y)\theta(JZ)\,[\theta(W)\theta(JU)-\theta(W)\theta(JU)]\\
&-2a\epsilon fg(X,JY)\theta(Z)\,[\theta(W)\theta(JU)-\theta(W)\theta(JU)]\\
&-2a\epsilon fg(X,Z)\theta(JY)\,[\theta(W)\theta(JU)-\theta(W)\theta(JU)]\\
&+2a\epsilon fg(X,JZ)\theta(Y)\,[\theta(W)\theta(JU)-\theta(W)\theta(JU)]\\
&+2a\epsilon fg(X,W)\theta(JU)\,[\theta(Y)\theta(JZ)-\theta(Z)\theta(JY)]\\
&-2a\epsilon fg(X,JW)\theta(U)\,[\theta(Y)\theta(JZ)-\theta(Z)\theta(JY)]\\
&-2a\epsilon fg(X,U)\theta(JW)\,[\theta(Y)\theta(JZ)-\theta(Z)\theta(JY)]\\
&+2a\epsilon fg(X,JU)\theta(W)\,[\theta(Y)\theta(JZ)-\theta(Z)\theta(JY)]\\
&+g(\xi,Y)R_{XZWU}+\epsilon g(\xi,JY)R_{JXZWU}+g(\xi,Z)R_{YXWU}\\
&+\epsilon g(\xi,JZ)R_{YJXWU}+g(\xi,W)R_{YZXU}+\epsilon g(\xi,JW)R_{YZJXU}\\
&+g(\xi,U)R_{YZWX}+\epsilon g(\xi,JU)R_{YZWJX},
\end{aligned}$$

and making use of (8.2) and (8.3) we obtain

$$\begin{aligned}
(\nabla_X R)_{YZWU} = -a\epsilon f\,\{&-2g(X,Y)\theta(JZ)\,[\theta(W)\theta(JU)-\theta(JW)\theta(U)]\\
&+2g(X,JY)\theta(Z)\,[\theta(W)\theta(JU)-\theta(JW)\theta(U)]\\
&+2g(X,Z)\theta(JY)\,[\theta(W)\theta(JU)-\theta(JW)\theta(U)]\\
&-2g(X,JZ)\theta(Y)\,[\theta(W)\theta(JU)-\theta(JW)\theta(U)]\\
&-2g(X,W)\theta(JU)\,[\theta(Y)\theta(JZ)-\theta(Z)\theta(JY)]\\
&+2g(X,JW)\theta(U)\,[\theta(Y)\theta(JZ)-\theta(Z)\theta(JY)]\\
&+2g(X,U)\theta(JW)\,[\theta(Y)\theta(JZ)-\theta(Z)\theta(JY)]\\
&-2g(X,JU)\theta(W)\,[\theta(Y)\theta(JZ)-\theta(Z)\theta(JY)]\}
\end{aligned}$$

$$
\begin{aligned}
&- g(\xi, X)R_{ZYWU}\\
&- a\epsilon f\{-2g(X,JY)\theta(Z)[\theta(W)\theta(JU) - \theta(U)\theta(JW)]\\
&- 2g(Y,JZ)\theta(X)[\theta(W)\theta(JU) - \theta(U)\theta(JW)]\\
&- 2g(Z,JX)\theta(Y)[\theta(W)\theta(JU) - \theta(U)\theta(JW)]\\
&+ g(X,W)\theta(JU)[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)]\\
&+ g(Y,W)\theta(JU)[\theta(Z)\theta(JX) - \theta(X)\theta(JZ)]\\
&+ g(Z,W)\theta(JU)[\theta(X)\theta(JY) - \theta(Y)\theta(JX)]\\
&- g(X,JW)\theta(U)[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)]\\
&- g(Y,JW)\theta(U)[\theta(Z)\theta(JX) - \theta(X)\theta(JZ)]\\
&- g(Z,JW)\theta(U)[\theta(X)\theta(JY) - \theta(Y)\theta(JX)]\\
&- g(X,U)\theta(JW)[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)]\\
&- g(Y,U)\theta(JW)[\theta(Z)\theta(JX) - \theta(X)\theta(JZ)]\\
&- g(Z,U)\theta(JW)[\theta(X)\theta(JY) - \theta(Y)\theta(JX)]\\
&+ g(X,JU)\theta(W)[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)]\\
&+ g(Y,JU)\theta(W)[\theta(Z)\theta(JX) - \theta(X)\theta(JZ)]\\
&+g(Z,JU)\theta(W)[\theta(X)\theta(JY) - \theta(Y)\theta(JX)]\}\\
&- g(\xi, X)R_{ZYWU}\\
&- a\epsilon f\{2g(X,Y)\theta(JZ)[\theta(W)\theta(JU) - \theta(U)\theta(JW)]\\
&- 2g(Y,JZ)\theta(X)[\theta(W)\theta(JU) - \theta(U)\theta(JW)]\\
&- 2g(Z,X)\theta(JY)[\theta(W)\theta(JU) - \theta(U)\theta(JW)]\\
&+ g(X,W)\theta(JU)[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)]\\
&+ g(JY,W)\theta(JU)[\theta(JZ)\theta(JX) + \theta(X)\theta(Z)]\\
&+ g(JZ,W)\theta(JU)[-\theta(X)\theta(Y) - \theta(JY)\theta(JX)]\\
&- g(X,JW)\theta(U)[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)]\\
&- g(Y,W)\theta(U)[\theta(JZ)\theta(JX) + \theta(X)\theta(Z)]\\
&- g(Z,W)\theta(U)[-\theta(X)\theta(Y) - \theta(JY)\theta(JX)]\\
&- g(X,U)\theta(JW)[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)]\\
&- g(JY,U)\theta(JW)[\theta(JZ)\theta(JX) + \theta(X)\theta(Z)]\\
&- g(JZ,U)\theta(JW)[-\theta(X)\theta(Y) - \theta(JY)\theta(JX)]\\
&+ g(X,JU)\theta(W)[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)]\\
&+ g(Y,U)\theta(W)[\theta(JZ)\theta(JX) + \theta(X)\theta(Z)]\\
&+g(Z,U)\theta(W)[-\theta(X)\theta(Y) - \theta(JY)\theta(JX)]\}\\
&- g(\xi, X)R_{UWYZ}\\
&- a\epsilon f\{-2g(X,JW)\theta(U)[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)]\\
&- 2g(W,JU)\theta(X)[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)]\\
&- 2g(U,JX)\theta(W)[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)]\\
&+ g(X,Y)\theta(JZ)[\theta(W)\theta(JU) - \theta(U)\theta(JW)]\\
&+ g(Y,W)\theta(JZ)[\theta(U)\theta(JX) - \theta(X)\theta(JU)]\\
&+ g(U,Y)\theta(JZ)[\theta(X)\theta(JW) - \theta(W)\theta(JX)]\\
&- g(X,JY)\theta(Z)[\theta(W)\theta(JU) - \theta(U)\theta(JW)]\\
&- g(W,JY)\theta(Z)[\theta(U)\theta(JX) - \theta(X)\theta(JU)]\\
&- g(U,JY)\theta(Z)[\theta(X)\theta(JW) - \theta(W)\theta(JX)]\\
&- g(X,Z)\theta(JY)[\theta(W)\theta(JU) - \theta(U)\theta(JW)]\\
&- g(W,Z)\theta(JY)[\theta(U)\theta(JX) - \theta(X)\theta(JU)]
\end{aligned}
$$

$$
\begin{aligned}
&- g(Z,U)\theta(JY)\left[\theta(X)\theta(JW) - \theta(W)\theta(JX)\right]\\
&+ g(X,JZ)\theta(Y)\left[\theta(W)\theta(JU) - \theta(U)\theta(JW)\right]\\
&+ g(W,JZ)\theta(Y)\left[\theta(U)\theta(JX) - \theta(X)\theta(JU)\right]\\
&+g(U,JZ)\theta(Y)\left[\theta(X)\theta(JW - \theta(W)\theta(JX)\right]\}\\
&- g(\xi,X)R_{UWYZ}\\
&- a\epsilon f\left\{2g(X,W)\theta(JU)\left[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)\right]\right.\\
&- 2g(W,JU)\theta(X)\left[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)\right]\\
&- 2g(U,X)\theta(JW)\left[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)\right]\\
&+ g(X,Y)\theta(JZ)\left[\theta(W)\theta(JU) - \theta(U)\theta(JW)\right]\\
&+ g(Y,JW)\theta(JZ)\left[\theta(JU)\theta(JX) + \theta(X)\theta(U)\right]\\
&+ g(JU,Y)\theta(JZ)\left[-\theta(X)\theta(W) - \theta(JW)\theta(JX)\right]\\
&- g(X,JY)\theta(Z)\left[\theta(W)\theta(JU) - \theta(U)\theta(JW)\right]\\
&- g(W,Y)\theta(Z)\left[\theta(JU)\theta(JX) + \theta(X)\theta(U)\right]\\
&- g(U,Y)\theta(Z)\left[-\theta(X)\theta(W) - \theta(JW)\theta(JX)\right]\\
&- g(X,Z)\theta(JY)\left[\theta(W)\theta(JU) - \theta(U)\theta(JW)\right]\\
&- g(JW,Z)\theta(JY)\left[\theta(JU)\theta(JX) + \theta(X)\theta(U)\right]\\
&- g(Z,JU)\theta(JY)\left[-\theta(X)\theta(W) - \theta(JW)\theta(JX)\right]\\
&+ g(X,JZ)\theta(Y)\left[\theta(W)\theta(JU) - \theta(U)\theta(JW)\right]\\
&+ g(W,Z)\theta(Y)\left[\theta(JU)\theta(JX) + \theta(X)\theta(U)\right]\\
&+ g(U,Z)\theta(Y)\left[-\theta(X)\theta(W) - \theta(JW)\theta(JX)\right].
\end{aligned}
$$

Rearranging the terms the previous formula becomes

$$
\begin{aligned}
(\nabla_X R)_{YZWU} &= 4g(\xi,X)R_{YZWU}\\
&- 2a\epsilon f\theta(X)\left\{-g(Y,W)\left[\theta(U)\theta(Z) + \theta(JU)\theta(JZ)\right]\right.\\
&+ g(Y,U)[\theta(JW)\theta(JZ) + \theta(W)\theta(Z)]\\
&+ g(Z,W)\left[\theta(U)\theta(Y) + \theta(JU)\theta(JY)\right]\\
&- g(Z,U)[\theta(JW)\theta(JY) + \theta(W)\theta(Y)]\\
&+ g(Y,JW)\left[\theta(U)\theta(JZ) - \theta(JU)\theta(Z)\right]\\
&- g(Y,JU)[\theta(W)\theta(JZ) - \theta(JW)\theta(Z)]\\
&- g(Z,JW)\left[\theta(U)\theta(JY) - \theta(JU)\theta(Y)\right]\\
&+ g(Z,JU)[\theta(W)\theta(JY) - \theta(JW)\theta(Y)]\\
&- 2g(Y,JZ)\left[\theta(W)\theta(JU) - \theta(U)\theta(JW)\right]\\
&\left.-2g(W,JU)\theta(X)\left[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)\right]\right\}\\
&- a\epsilon f\left\{2g(X,Y)\theta(JZ)\left[\theta(W)\theta(JU) - \theta(U)\theta(JW)\right]\right.\\
&- 2g(X,Z)\theta(JY)\left[\theta(W)\theta(JU) - \theta(U)\theta(JW)\right]\\
&+ 2g(X,W)\theta(JU)\left[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)\right]\\
&- 2g(X,U)\theta(JW)\left[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)\right]\\
&- 2g(X,JY)\theta(Z)\left[\theta(W)\theta(JU) - \theta(U)\theta(JW)\right]\\
&+ 2g(X,JZ)\theta(Y)\left[\theta(W)\theta(JU) - \theta(U)\theta(JW)\right]\\
&- 2g(X,JW)\theta(U)\left[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)\right]\\
&\left.+2g(X,JU)\theta(W)\left[\theta(Y)\theta(JZ) - \theta(Z)\theta(JY)\right]\right\},
\end{aligned}
$$

that is,

$$
\nabla_X R = 4\theta(X) \otimes (R - \frac{1}{2}ag \boxtimes r) - 2a\epsilon \Big((X^\flat \wedge (\theta \circ J)) \odot \rho + (JX^\flat \wedge (\theta)) \odot \rho\Big),
$$

where ρ is the Ricci form and $\boxtimes$ stands for the ϵ-complex Kulkarni-Nomizu product defined as

$$\begin{aligned} h \boxtimes k(X_1, X_2, X_3, X_4) &= h(X_1, X_3)k(X_2, X_4) + h(X_2, X_4)k(X_1, X_3) \\ &- h(X_1, X_4)k(X_2, X_3) - h(X_2, X_3)k(X_1, X_4) \\ &- \epsilon h(X_1, JX_3)k(X_2, JX_4) - \epsilon h(X_2, JX_4)k(X_1, JX_3) \\ &+ \epsilon h(X_1, JX_4)k(X_2, JX_3) + \epsilon h(X_2, JX_3)k(X_1, JX_4) \\ &- 2\epsilon h(X_1, JX_2)k(X_3, JX_4) - 2\epsilon h(X_3, JX_4)k(X_1, JX_2), \end{aligned}$$

for h and k symmetric $(0, 2)$-tensors.

Bibliography

[1] E. Abbena, S. Garbiero, *Almost hermitian homogeneous structures*, Proc. Edinb. Math. Soc. **31** (2) (1988), 375–395.

[2] D.V. Alekseevsky, V. Cortés, *Classification of pseudo-Riemannian symmetric spaces of quaternionic Kähler type*, Lie Groups and Invariant Theory, Amer. Math. Soc. Transl. (2) 213, Amer. Math. Soc., Providence, RI., 2005, 33–62.

[3] D.V. Alekseevsky, C. Medori, A. Tomassini, *Homogeneous para-Kähler Einstein manifolds*, arXiv:0806.2272v2.

[4] W. Ambrose, I.M. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. **25** (1958), 647–669.

[5] H. Bacry, *Lecons sur la théorie des groupes et les symétries des particles élémentaires*, Gordon and Breach, Paris, 1967.

[6] M. Barros, A. Romero, *Indefinite Kähler Manifolds*, Math. Ann. **261** (1982), 55–62.

[7] A.O. Barut, R. Rączka, *Theory of group Representations and Applications*, Polish Scientific Publishers, Warszawa (1977).

[8] W. Batat, P.M. Gadea, J.A. Oubiña, *Homogeneous pseudo-Riemannian structures of linear type*, J. Geom. Phys. **61** (2011), 745–764.

[9] M. Berger, Sur les grupes d'holonomie des variétés à connexion affine et des variétés riemanniennes, Bull. Soc. Math. France **83** (1955), 279–330.

[10] M. Berger, *Les espaces symétriques non compacts*, Ann. Sci. École Norm. Sup. **74** (1957), 85–177.

[11] A.L. Besse, *Einstein Manifolds.* Springer-Verlag, Berlin Heidelberg (1987).

[12] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics 203, Birkhäuser Boston, Inc., Boston, MA (2002).

[13] A. Borel, A. Lichnerowicz, *Groupes d'holonomie des variétés riemanniennes*, C. R. Acad. Sci. Paris. **234** (1952), 279–300.

[14] R. Bryant, *Classical, exceptional, and exotic holonomies: a status report*, in Actes de la Table Ronde de Géométrie Différentielle en l'Honneur de Marcel Berger, Soc. Math. France (1996), 93–166.

[15] M. Cahen, N. Wallach, *Lorentzian symmetric spaces*, Bull. Amer. Math. Soc. **76** 585–591 (1970).

[16] M. Castrillón López, P. M. Gadea, A.F. Swann, *Homogeneous quaternionic Kähler structures and quaternionic hyperbolic space*, Transformation Groups, Vol. **11**, No. **4** (2006), 576–608.

[17] M. Castrillón López, P.M. Gadea, A.F. Swann, *Homogeneous structures on real and complex hyperbolic spaces*, Illinois J. Math. **53** (2) (2009), 561–574.

[18] M. Castrillón López, I. Luján, *Strongly degenerate homogeneous pseudo-Kähler structures of linear type and complex plane waves*, J. Geom. Phys. **73** (2013), 1–19.

[19] M. Castrillón López, I. Luján, *Reduction of Homogeneous Riemannian Structures*, to appear in Proc. Edinb. Math. Soc.

[20] M. Castrillón López, I. Luján, *Homogeneous structures of linear type on ϵ-Kähler and ϵ-quaternion Kähler manifolds*, arXiv:1310.4323.

[21] Q.S. Chi, S. Merkulov, L. Schwachhöfer, *On the existence of infinite series of exotic holonomies*, Invent. Math. **126** (1996), 391–411.

[22] D. Chinea, C. González, *A Classification of Almost Contact Metric Manifolds*, Ann. Mat. Pur. Appl. (IV) Vol. CLVI (1990), 15–36.

[23] V. Cruceanu, P. Fortuny, P. M. Gadea, *A Survey on Paracomplex Geometry*, Rocky Mt. J. Math. **26** (1) (1996), 83–115.

[24] G. DeRham, *Sur la réductibilité d'un espace de Riemann*, Comm. Math. Helv. **26** (1953), 328–344.

[25] M.E. Fels, A.G. Renner, *Non-reductive four dimensional homogeneous pseudo-Riemannian Manifolds*, Canadian J. Math., **58** (2) (2006), 282–311.

[26] A. Fino, *Intrinsic Torsion and Weak Holonomy*, Math. J. Toyama Univ. Vol. **21** (1998), 1–22.

[27] A. Fino, I. Luján, *Torsion-free $G^*_{2(2)}$-structures with full holonomy on nilmanifolds*, to appear in Adv. Geom.

[28] P.M. Gadea, A. Montesinos Amilibia, *Spaces of constant para-holomorphic sectional curvature*, Pac. J. Math. **136** (1) (1989), 85–101.

[29] P.M. Gadea, A. Montesinos Amilibia, J. Muñoz Masqué, *Characterizing the complex hyperbolic space by Kähler homogeneous structures*, Math. Proc. Camb. Phil. Soc. **128** (1) (2000), 87–94.

[30] P.M. Gadea, J.A. Oubiña, *Homogeneous pseudo-Riemannian structures and homogeneous almost para-Hermitian structures*, Houston J. Math. **18** (1992), 449–465.

[31] P.M. Gadea, José A. Oubiña, *Reductive Homogeneous Pseudo-Riemannian manifolds*, Mh. Math. **124** (1997), 17–34.

[32] P.M. Gadea, J.A. Oubiña, *Homogeneous Riemannian Structures on Berger 3-spheres*, Proc. Edinb. Math. Soc. **48** (2005), 375–387.

[33] A. Galaev, *Holonomy groups and special geometric structures of pseudo-Kählerian manifolds of index* 2, Berlin: Humboldt-Univ., Mathematisch-Naturwissenschaftliche Fakultat II (Dissertation) (2006), 135 p.

[34] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press 1978.

[35] S. Ishihara, *Quaternion Kählerian manifolds and fibred Riemannian spaces with Sasakian 3-structure*, Koday Math. Sem. Rep. **25** (1973), 321-329.

[36] S. Ishihara, *Quaternion Kählerian manifolds*, J. Differ. Geom. **9** (1974), 483–500.

[37] V. F. Kiričenko, *On homogeneous Riemannian spaces with an invariant structure tensor*, Sov. Math., Dokl. **21** (1980), 734-737.

[38] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, John Wiley & Sons, Inc. (Interscience Division), New York, Volumes I & II (1963, 1969).

[39] O. Kowalski, *Counter-example to the "Second Singer's Theorem"*, Ann. Global Anal. Geom. **8** (1990), 211–214.

[40] O. Kowalski, F. Tricerri, *A canonical connection for locally homogeneous Riemmanian manifolds*, Proc. of the Conf. on Global Analysis and Global Differential Geometry, Berlin 1990, Lecture Notes in Math. n. 1441, Springer Verlag, Berlin, Heidelberg, New York 1991, 97–103.

[41] I. Luján, *Reductive locally homogeneous pseudo-Riemannian manifolds and Ambrose-Singer connections*, arXiv:1405.0826.

[42] I. Luján, A. Swann, *Non-degenerate homogeneous ϵ-Kähler and ϵ-quaternion Kähler structures of linear type*, arXiv:1311.3072.

[43] J.E. Marsden, J Ostrowski, *Symmetries in motion: geometric foundations of motion control*, Nonlinear Sci. Today, 21 pp. (1996).

[44] S. Merkulov, L. Schwachhöfer, *Classification of irreducible holonomies of torsion-free affine connections*, Ann. Math. **150** (1999), 77–149.

[45] P. Meessen, *Homogeneous Lorentzian spaces admitting a homogeneous structure of type $\mathcal{T}_1 + \mathcal{T}_3$*. J. Geom. Phys. **56**, 754–761 (2006).

[46] A. Montesinos Amilibia, *Degenerate homogeneous structures of type $\mathcal{S}_1$ on pseudo-Riemannian manifolds*, Rocky Mt. J. Math. **31** (2) (2001), 561–579.

[47] R. Montgomery, *Isoholonomic problems and some applications*, Comm. Math. Phys. **128** (3) (1990), 565–592.

[48] A. Newlander, L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. **65** (1957), 391–404.

[49] L. Nicolodi, F. Tricerri, *On Two Theorems of I.M. Singer about Homogeneous Spaces*, Ann. Global Anal. Geom., **8** (2) (1990), 193–209.

[50] K. Nomizu, *On local and global existence of Killing vector fields*, Ann. Math. **72** (1960), 105–120.

[51] K. Ogiue, *On fiberings of almost contact manifolds*, Ködai Math. Sem. Rep. **17** (1965) 53–62.

[52] J.D. Pérez, F.G. Santos, *Indefinite quaternion space forms*, Ann. Mat. Pur. Appl. **132** (1) (1982), 383–398.

[53] S.M. Salamon, *Riemannian Geometry and Holonomy Groups*, Pitman Research Notes Math. 201, Longman (1989).

[54] K. Sekigawa, *Notes on homogeneous almost hermitian manifolds*, Hokkaido Math. J. **7** (1978), 206-213.

[55] José M.M. Senovilla, *Singularity Theorems and Their Consequences*, Gen. Relat. Gravit. **30** (5) (1998), 701–848.

[56] J. Simons, *On the transitivity of holonomy systems*, Ann. Math. **76** (2) (1962), 213–234.

[57] A. Spiro, *Lie Pseudogroups and Locally Homogeneous Riemannian Spaces*, Boll. UMI **6-B** (1992), 843–872.

[58] T. Takahashi, *Sasakian manifolds with pseudo Riemannian metric*, Tohoku Math. Journ. **21** (1969), 271–290.

[59] F. Tricerri, *Locally homogeneous Riemannian manifolds*, Rend. Sem. Mat. Univ. Politec. Torino **50** (4) (1992), 411–426.

[60] F. Tricerri, L. Vanhecke, *Homogeneous Structures on Riemannian Manifolds*, Cambridge University Press, Cambridge, (1983).

[61] F. Tricerri, Y. Watanabe, *Infinitesimal Models and Locally Homogeneous Almost Hermitian Manifolds*, Math. J. Toyama Univ. **18** (1995), 147–154.

[62] S. Vukmirovic, *Para-quaternionic reduction*, arXiv:math/0304424.

[63] H.C. Wang, *On invariant connections over a principal fibre bundle.* Nagoya Math. J. **13** (1958), 1–19.

[64] H. Weyl, *The Classical Groups*, Princeton University Press, 1939.

[65] H. Wu, *Holonomy groups of indefinite metrics*, Pac. J. Math. **20** (1967), 351–382.